# Theory of wind-driven circulation

## The classical view

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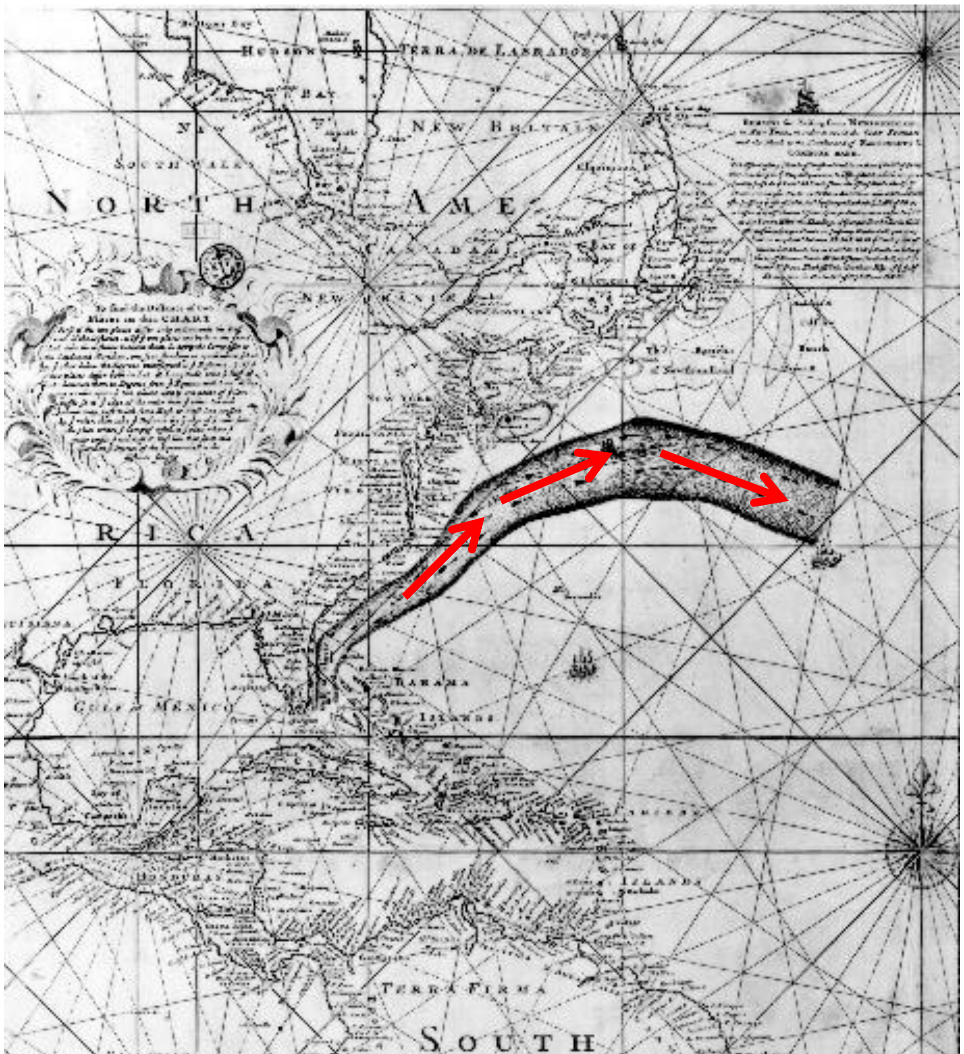
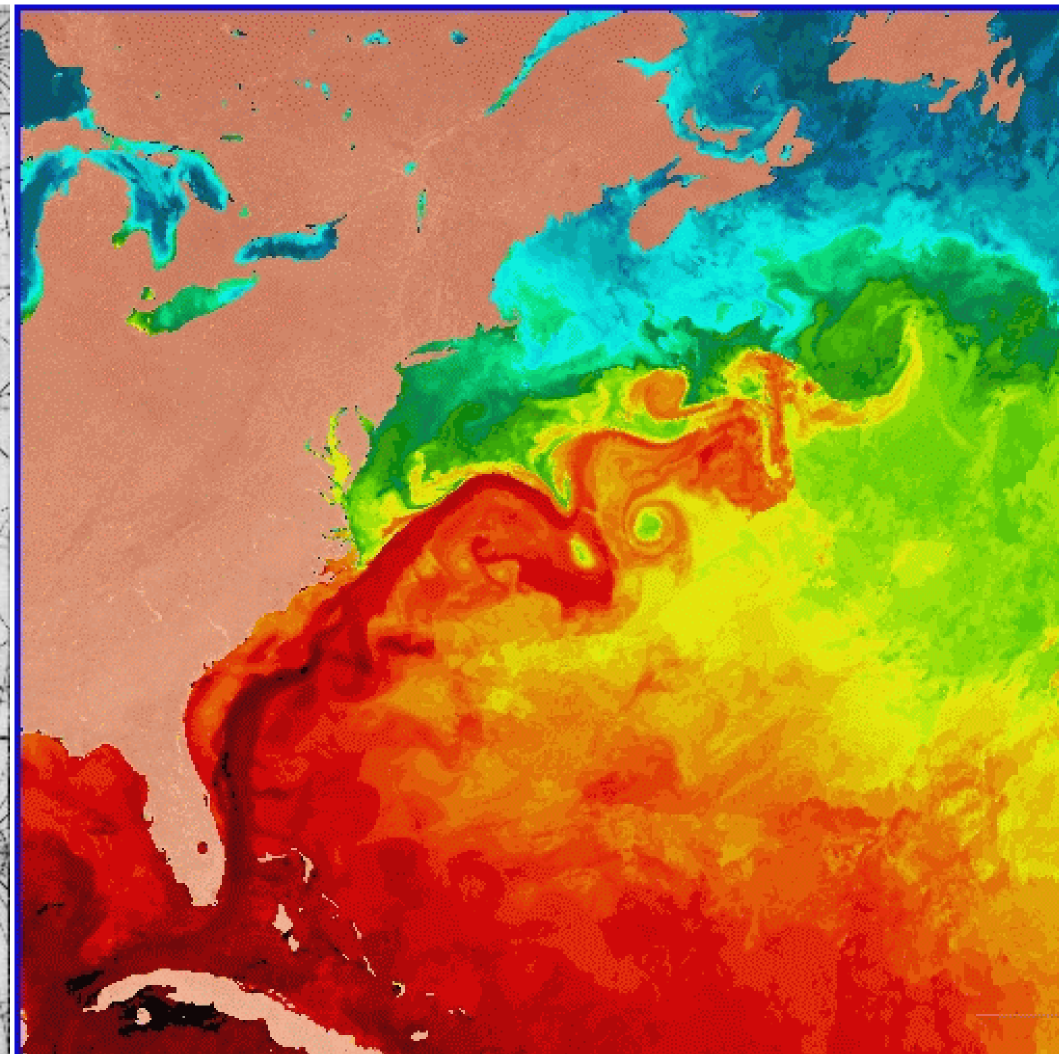
# Wind-driven circulation has been studied for a long time

**Gulf Stream** from Benjamin Franklin's map (widely used by ship captains) and SST map from satellite



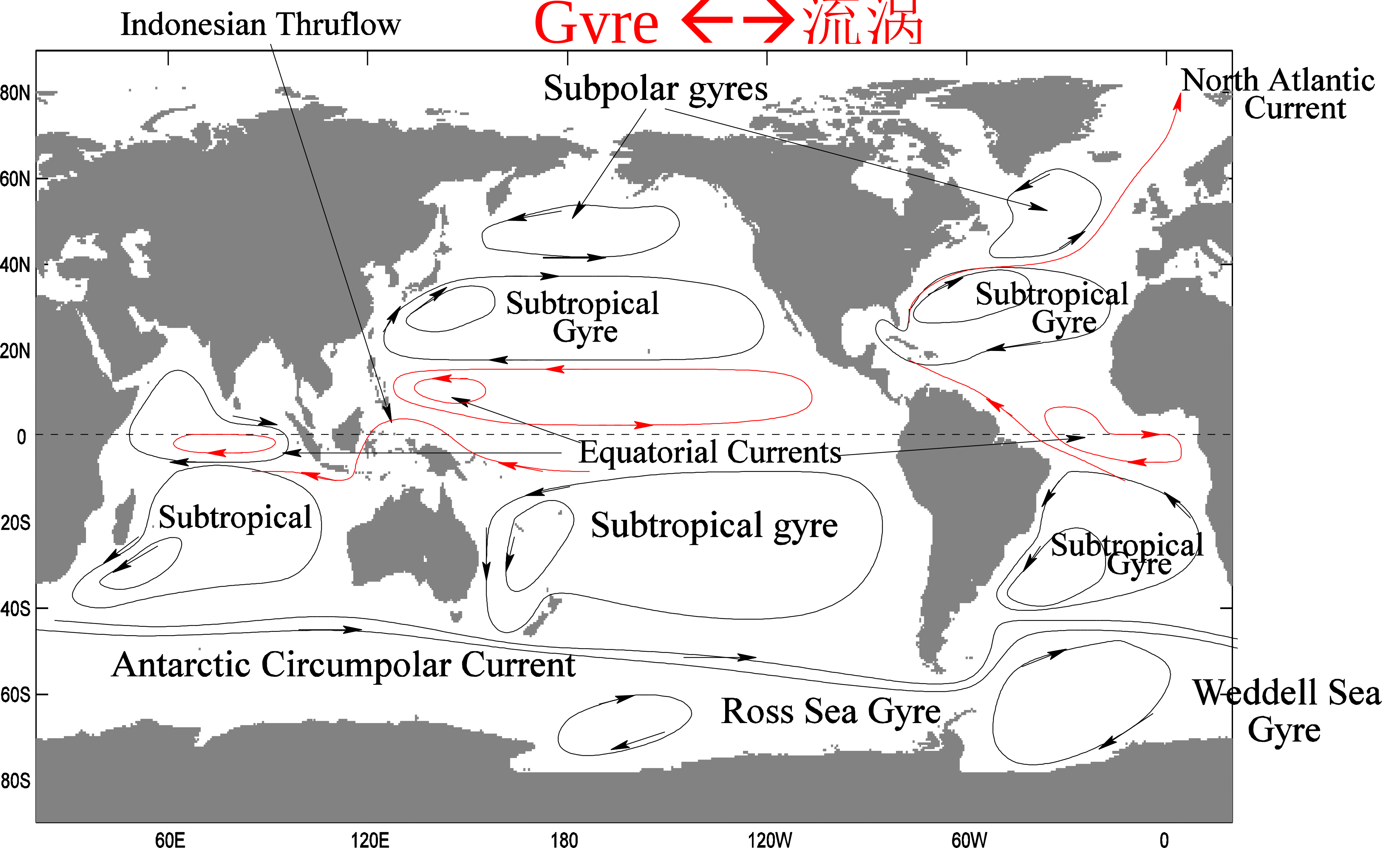
Richardson, Science (1980)

SST satellite image (U. Miami RSMAS)

# Major wind-driven gyres in the world oceans

## Why gyres? Why western boundary currents?

Gyre ←→流涡



# The Ekman Spiral (Ekman, 1905)

- Momentum equations

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(A\frac{\partial v}{\partial z}\right)$$

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(A\frac{\partial u}{\partial z}\right)$$

$$A\frac{\partial u}{\partial z} = \tau^x/\rho,\ A\frac{\partial v}{\partial z} = \tau^y/\rho \quad at\ z = 0$$

$$(u,v) \to 0,\ \ as\ z \to -\infty$$

- Velocity decomposition

$$u = U_g + u_e = -\frac{1}{\rho}\frac{\partial p}{\partial y} + u_e \qquad v = V_g + v_e = \frac{1}{\rho}\frac{\partial p}{\partial x} + v_e$$

- Ekman layer is thin (~30m), so the vertical shear of the geostrophic velocity is negligible

- Equations for the ageostrophic velocity

$$-fv_e = \frac{\partial}{\partial z}\left(A\frac{\partial u_e}{\partial z}\right),\ \ fu_e = \frac{\partial}{\partial z}\left(A\frac{\partial v_e}{\partial z}\right)$$

# The Ekman transport

- Equations for the ageostrophic velocity

$$-fv_e = \frac{\partial}{\partial z}\left(A\frac{\partial u_e}{\partial z}\right),\ fu_e = \frac{\partial}{\partial z}\left(A\frac{\partial v_e}{\partial z}\right)$$

- Vertical integration leads to $-f\int_{-\infty}^{0} v_e dz = A\frac{\partial u_e}{\partial z}|_{z=0} = \tau^x/\rho$

$$-fV_e = \tau^x/\rho$$
$$fU_e = \tau^y/\rho$$

$$\Rightarrow \quad \vec{U}_e = -\vec{z}\times\vec{\tau}/f\rho$$

$$\tau = 0.1\,Pa,\ f = 10^{-4}/s \Rightarrow V_e = \tau/f\rho \square 1m^2/s$$

$$Assume\ layer\ 20m\ thick \Rightarrow v_e \square\ 5cm/s$$

$$V_e L = 6\times10^6 m^3/s = 6Sv$$

# The Ekman spiral

- Equations for the ageostrophic velocity

$$M = u_e + iv_e \qquad \frac{d^2M}{dz^2} - i\frac{f}{A}M = 0$$

$$M = c_1 e^{\lambda z} + c_2 e^{-\lambda z}, \ \lambda = \frac{\sqrt{2}}{2}(1+i)\sqrt{\frac{f}{A}}$$

- Applying boundary conditions

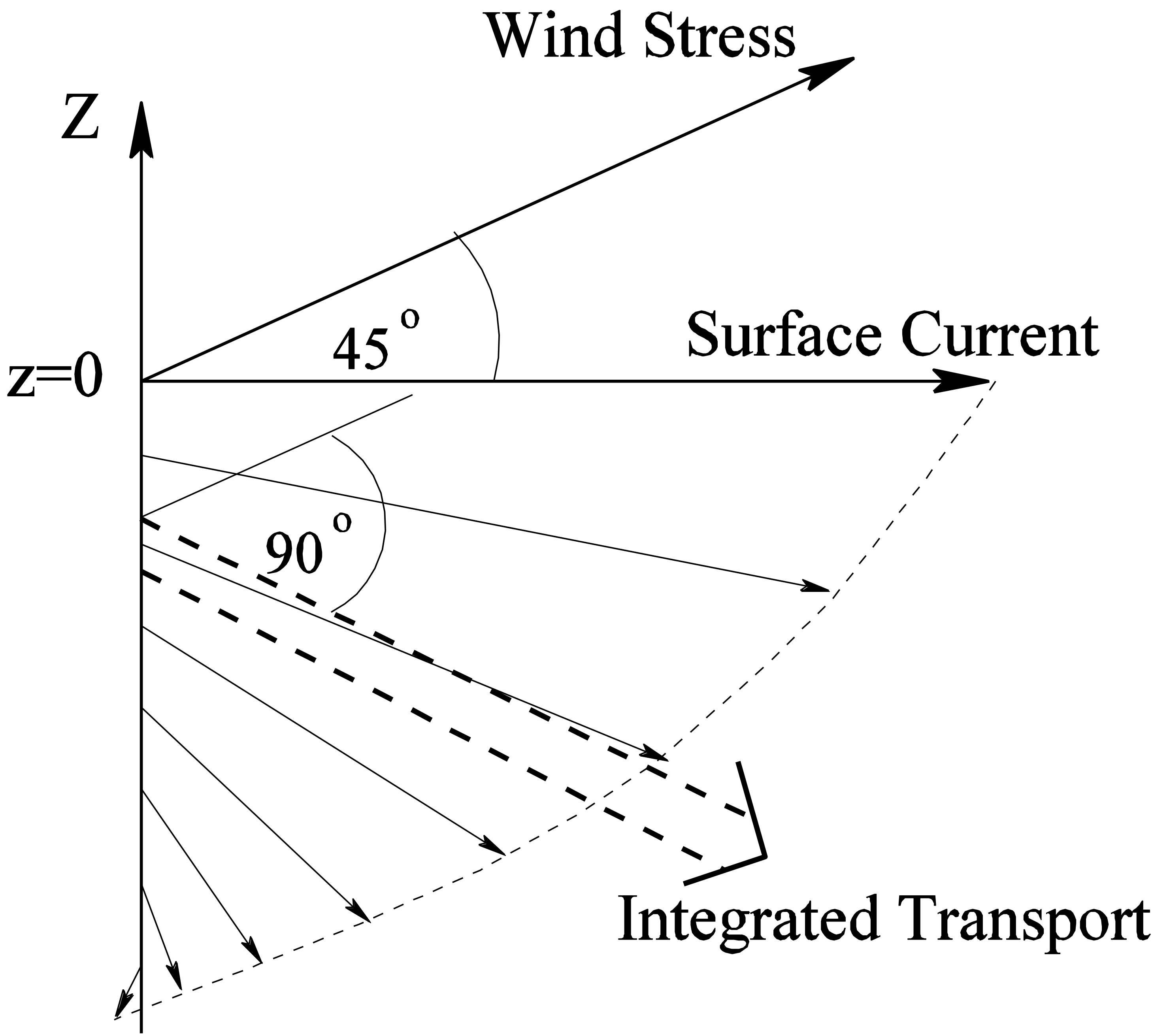
$$M \to 0 \ as\, z \to -\infty \Rightarrow c_2 = 0$$

$$\left.\frac{dM}{dz}\right|_{z=0} = \lambda c_1 = \frac{1}{\rho A}\left(\tau^x + i\tau^y\right) \qquad c_1 = \frac{1-i}{\sqrt{2fA\rho}}(\tau^x + i\tau^y)$$

- Ekman spiral

$$u_e + iv_e = \frac{1-i}{\sqrt{2fA\rho}}(\tau^x + i\tau^y)Exp\left[\frac{\sqrt{2}}{2}(1+i)\sqrt{\frac{f}{A}}z\right]$$

# The Ekman Spiral



# Consequence of Ekman convergence/divergence

- 1) In the ocean interior, Ekman convergence leads to Ekman pumping which drives the wind-driven gyres in the oceans
- 2) Off-shore (on-shore) Ekman flux leads to coastal upwelling (downwelling)
- 3) Off-equator Ekman flux induces equatorial upwelling
- 4) Ekman divergence leads to the upwelling associated with ACC --- the most important upwelling system in the world oceans

# The Ekman pumping (泵压/泵吸)

- Integrating continuity equation

$$u_x + v_y + w_z = 0$$

leads to

$$w_e = \int_{-H}^{0} \left(u_x + v_y\right) dz = \frac{\partial}{\partial x}\int_{-H}^{0} u_e dz + \frac{\partial}{\partial y}\int_{-H}^{0} v_e dz$$

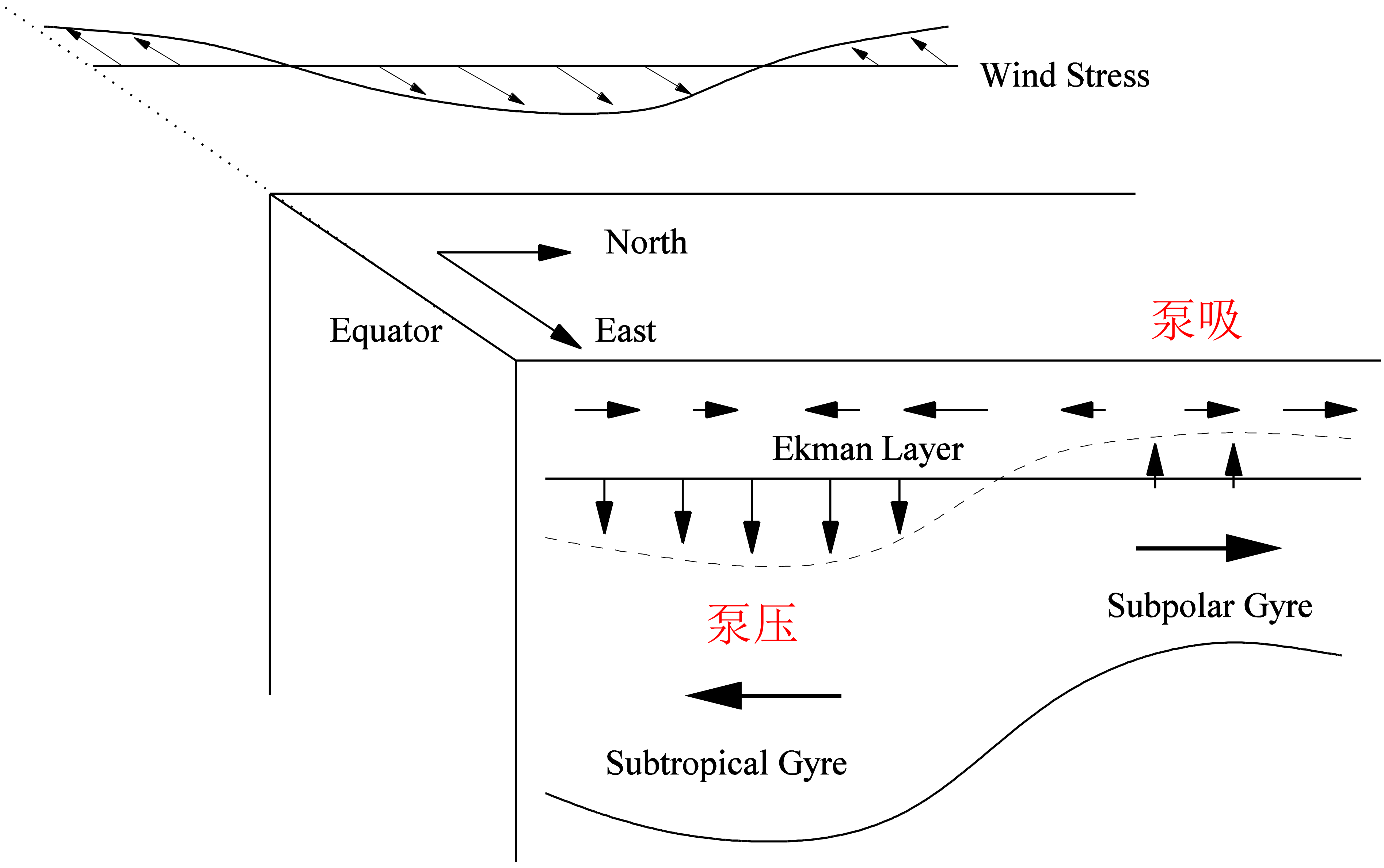
- Ekman pumping rate

$$w_e = \frac{\partial}{\partial x}\left(\frac{\tau^y}{f\rho}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f\rho}\right) = \frac{1}{f\rho}\left[\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right] + \frac{\beta \tau^x}{f^2 \rho}$$

- Typical values:

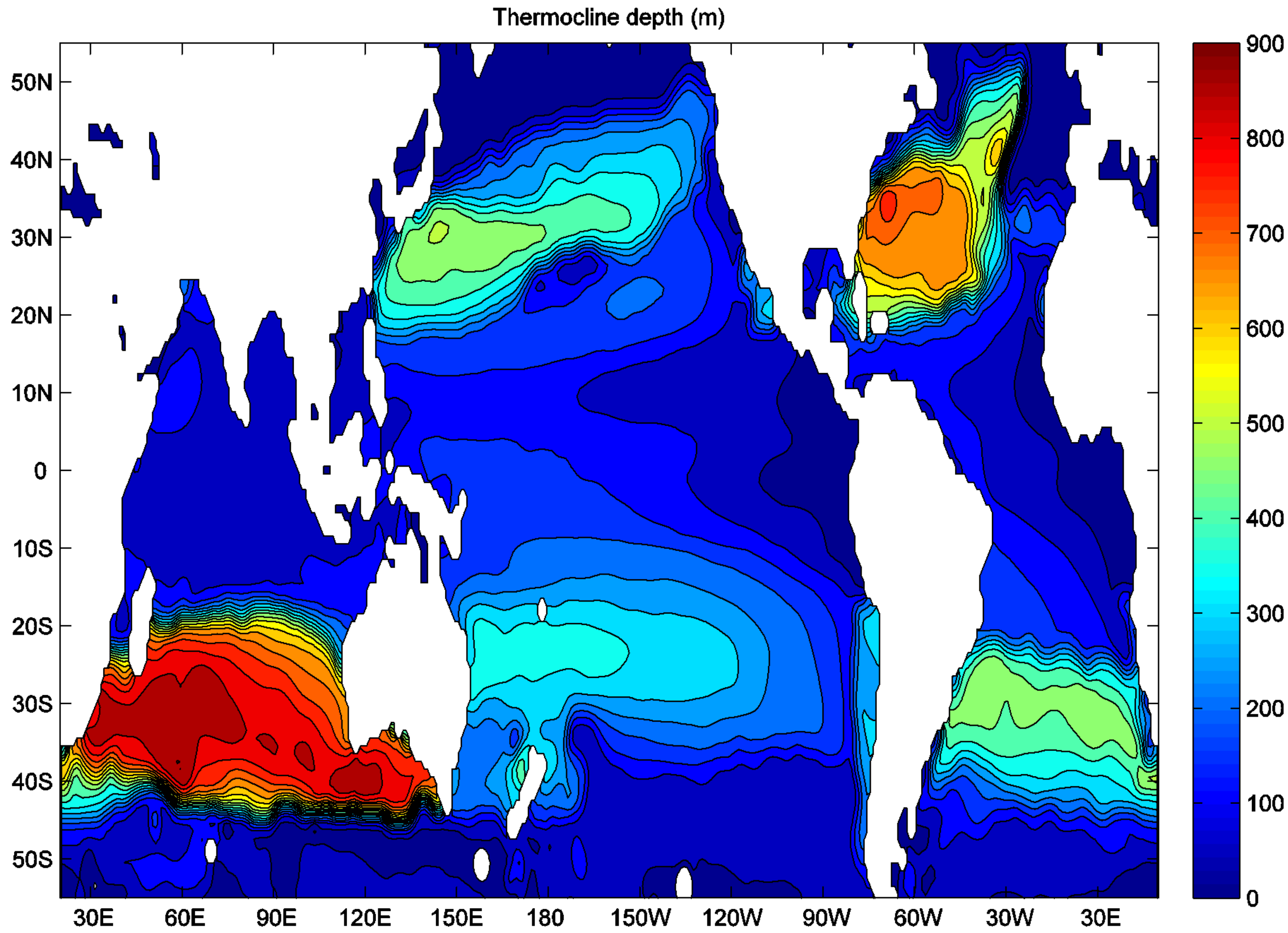
$$w_e \sim \mp 10^{-6}\, m/s$$

# Sketch of the wind-driven gyres

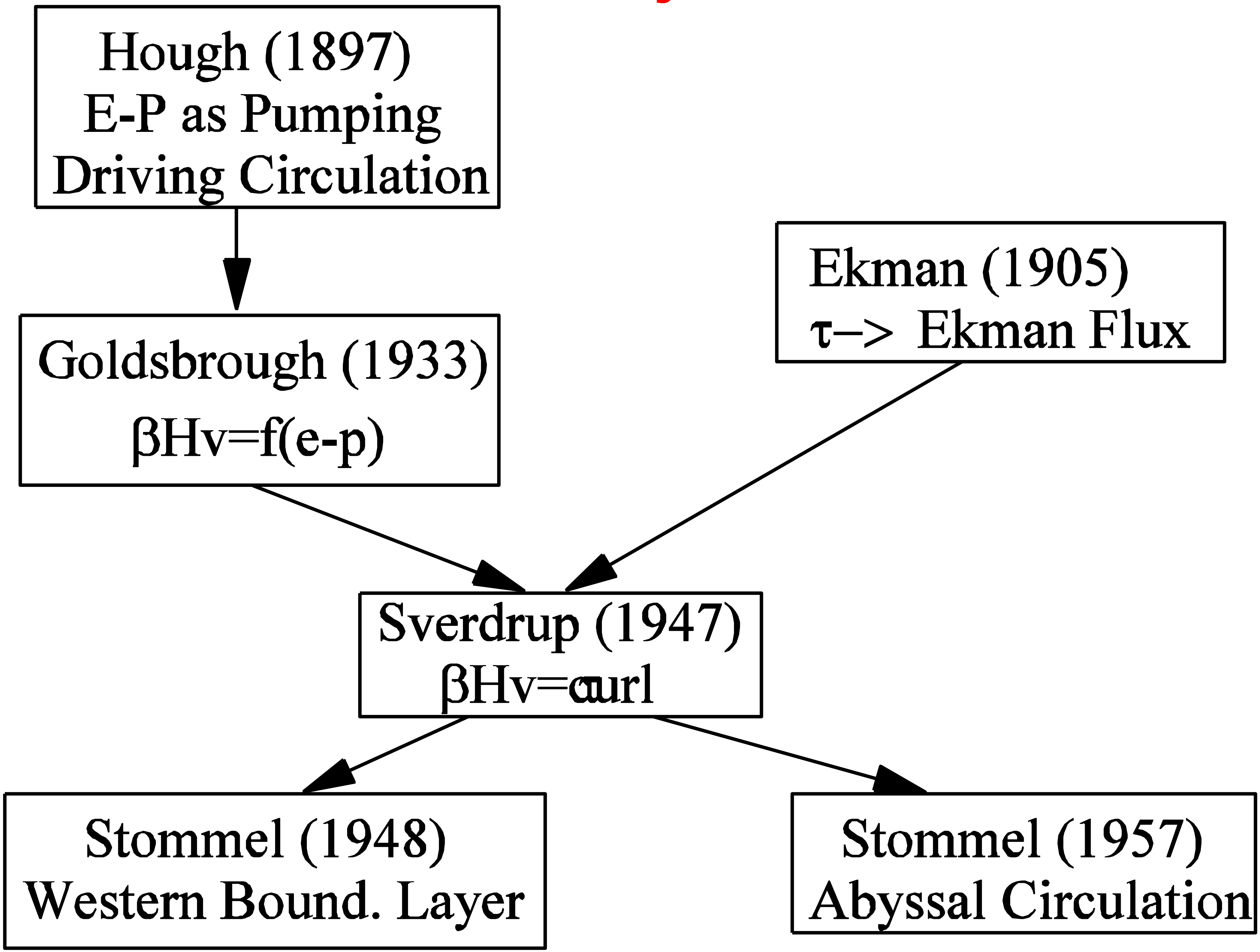


# Thermocline in the world oceans

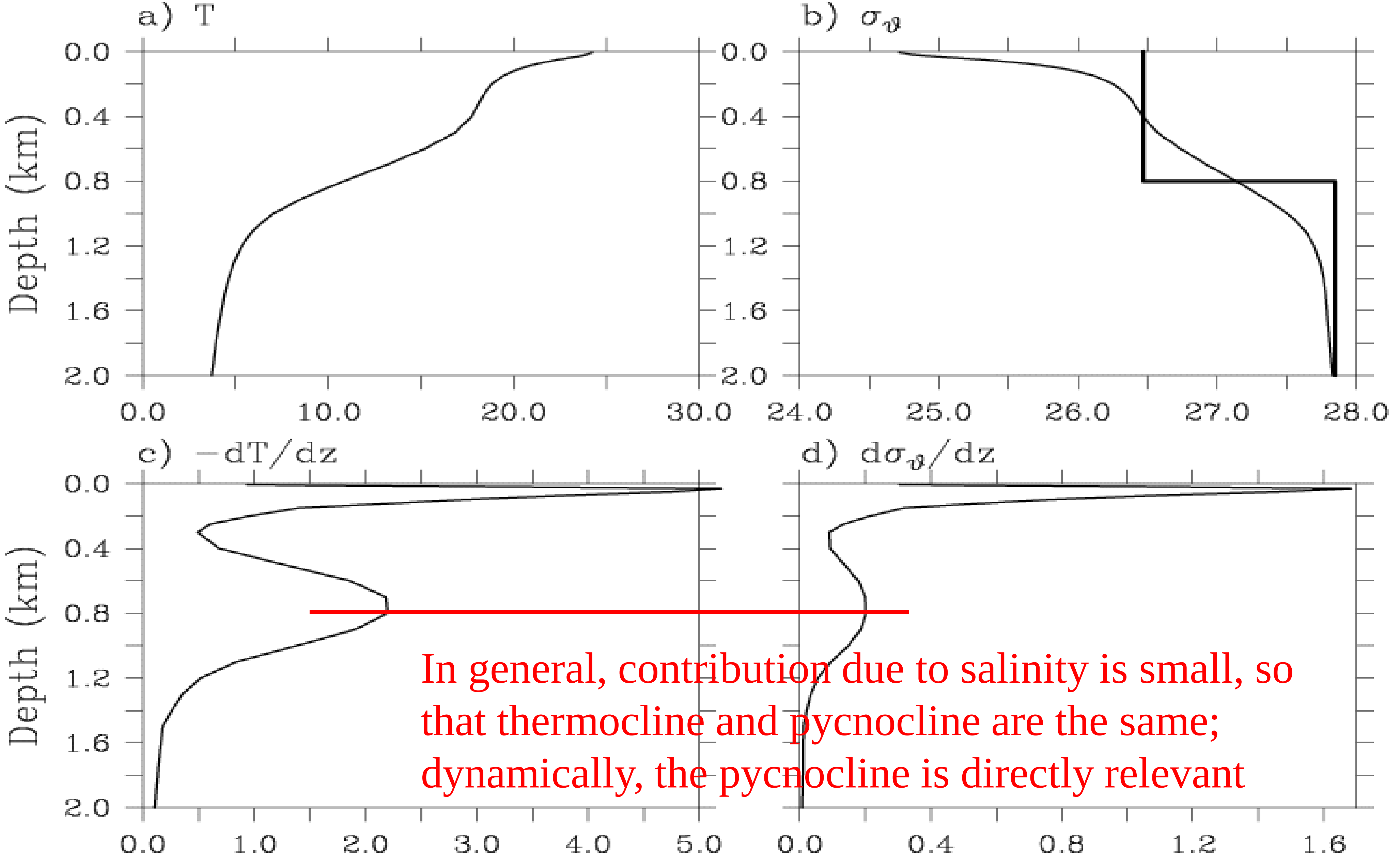
The best way to search for the thermocline is from the bottom and move upward



# Theoretical framework of one-layer models



# Thermocline at a station; The meaning of a reduced gravity model




In general, contribution due to salinity is small, so that thermocline and pycnocline are the same; dynamically, the pycnocline is directly relevant

# Pressure terms

- Rigid-lid approximation

 We should not take the so-called rigid-lid approximation as a “钢盖”。 This is just an approximation in which the upper surface of the ocean is set at the z=0 level with a suitable boundary condition.

- Pressure gradient under the rigid-lid approximation

Note that pressure at the z=0 is not constant, and this term can be eliminated based on certain assumptions

- Assume $p=p_a$ at z=0 and integrated the hydrostatic relation downward

$$\frac{1}{\rho_2}\nabla_h p_2 \approx g\nabla_h \zeta - g'_1 \nabla_h h_1 \approx 0 \Rightarrow \frac{1}{\rho_1}\nabla_h p_1 = g'\nabla_h h_1, \zeta \approx \frac{\Delta\rho}{\rho} h_1$$

# A reduced gravity model

- Basic assumptions:

  Treat the main thermocline as a step function in density

  Assume the lower layer is very thick and motionless

  Assume motions in the upper layer is adiabatic

- Basic equations

$$hu_t + h(uu_x + vu_y) - fhv = -g'hh_x + \tau^x/\rho_0 + A\nabla_h^2(hu) - Ru$$

$$hv_t + h(uv_x + vv_y) + fhu = -g'hh_y + \tau^y/\rho_0 + A\nabla_h^2(hv) - Rv$$

Reduced gravity   wind stress

$$h_t + (hu)_x + (hv)_y = 0$$

lateral friction   bottom friction

# A slightly different form

$$u_t + uu_x - \left(f + v_x - u_y\right)v + vv_x = -g'h_x + \frac{\tau^x}{\rho_0 h} + A_h\left(u_{xx} + u_{yy}\right) - ru$$

$$u_t - \left(f + \zeta\right)v = -B_x + \frac{\tau^x}{\rho_0 h} + A_h\left(u_{xx} + u_{yy}\right) - ru,\ B = g'h + \frac{u^2 + v^2}{2}$$

$$v_t + uv_x + \left(f + v_x - u_y\right)u + uu_y = -g'h_y + \frac{\tau^y}{\rho_0 h} + A_h\left(v_{xx} + v_{yy}\right) - rv$$

$$v_t + \left(f + \zeta\right)u = -B_y + \frac{\tau^y}{\rho_0 h} + A_h\left(v_{xx} + v_{yy}\right) - rv,\ B = g'h + \frac{u^2 + v^2}{2}$$

## The final form

$$u_t - \left(f + \zeta\right)v = -B_x + \frac{\tau^x}{\rho_0 h} + A_h\left(u_{xx} + u_{yy}\right) - ru;$$

$$v_t + \left(f + \zeta\right)u = -B_y + \frac{\tau^y}{\rho_0 h} + A_h\left(v_{xx} + v_{yy}\right) - rv;$$

# Mechanic energy balance

- Multiplying momentum equation by u and v and add:

$$h\vec{u}\cdot\nabla_h\left(\frac{u^2+v^2}{2}+g'h\right)=W$$

Along streamlines total energy change is balance by source/sink

Kinetic energy

Potential energy Bernoulli function

$$W=\frac{1}{\rho_0}(u\tau^x+v\tau^y)-R(u^2+v^2)+A[u\nabla_h{}^2(hu)+v\nabla_h{}^2(hv)]$$

Wind work positive source

Bottom friction sink

Lateral friction sink

Wind drive circulation is treated as adiabatic motions → thermodynamic is not included in such a theory

# Vorticity balance

- Cross-differentiating momentum equation and subtracting

$$\vec{u}\cdot\nabla_h q + q(u_x + v_y) = C$$

$$q = f + v_x - u_y \qquad C = \left(\frac{\tau^y}{\rho_0 h} - \frac{Rv}{h} + \frac{A}{h}\nabla_h{}^2(hv)\right)_x - \left(\frac{\tau^x}{\rho_0 h} - \frac{Ru}{h} + \frac{A}{h}\nabla_h{}^2(hu)\right)_y$$

Vorticity

Source of potential vorticity

Wind stress curl

Bottom friction sink

Lateral friction sink

$$h\vec{u}\cdot\nabla_h Q = C, \quad Q = \frac{f + v_x - u_y}{h}$$

# Potential vorticity balance

Along streamlines potential vorticity (PV) change is balance by source/sink

$$h\vec{u}\cdot\nabla_h Q = C,\ Q = \frac{f + v_x - u_y}{h}$$

A more accurate expression is to include the density jump

$$Q' = \left(f + v_x - u_y\right)\frac{\Delta\rho}{\rho_0 \Delta h}$$

For continuous stratification

$$Q' = -\left(f + v_x - u_y\right)\frac{\partial\rho}{\rho_0 \partial z}$$

For large-scale circulation, relative vorticity is small and negligible; thus, the commonly used forms are

$$Q' = f\frac{\Delta\rho}{\rho_0 \Delta h}\ or\ q' = \frac{f}{\Delta h} \qquad Q' = -\frac{f\,\partial\rho}{\rho_0 \partial z}$$

# Conservation laws

- For steady flow, we introduce a streamfunction

$$\psi_x = hv, \ \ \psi_y = -hu$$

- Two conservation laws

$$B = \frac{u^2 + v^2}{2} + g'h = F(\psi) \qquad Q = \frac{f + v_x - u_y}{h} = G(\psi)$$

- These two functions are not independent

Taking the gradient

$$\nabla B = \frac{dF}{d\psi} \nabla \psi$$

$$\frac{dF}{d\psi} = G(\psi), \ or \ \ Q\nabla\psi = \nabla B$$

# The interior solution, based on wind stress (Ekman flux is included, 全流)

$$-fhv = -g'hh_x + \tau^x / \rho_0$$

$$fhu = -g'hh_y$$

$$(hu)_x + (hv)_y = 0 \longrightarrow \psi_x = hv, \psi_y = -hu$$

- Cross-differentiation→

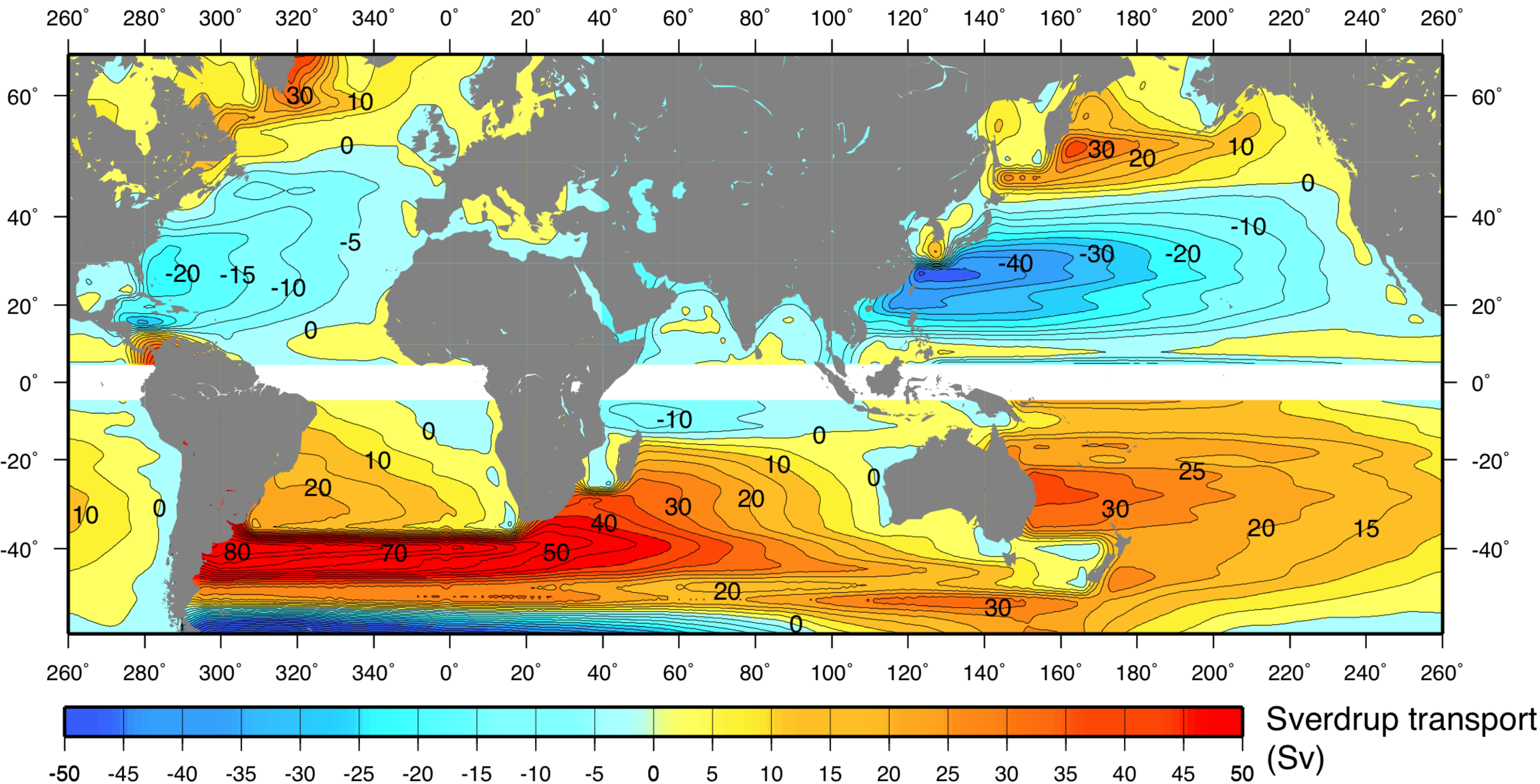
Sverdrup transport

$$\beta hv = -\tau_y^x / \rho_0 \longrightarrow \psi = \frac{1}{\rho_0 \beta} \tau_y^x (x_e - x)$$

$$hh_x = -\frac{f^2}{g'\rho_0\beta}\left(\frac{\tau^x}{f}\right)_y \longrightarrow h^2 = h_e^2 + \frac{2f^2}{g'\rho_0\beta}\left(\frac{\tau^x}{f}\right)_y (x_e - x)$$

Why integration from eastern boundary, not the western b.?

# Sverdrup transport in the world oceans




Descriptive Physical Oceanography Fig. 5.17

# The interior solution, based on Ekman pumping

(Ekman flux not included)

$$-fhv = -g'hh_x$$

$$fhu = -g'hh_y$$

$$(hu)_x + (hv)_y = -w_e,\ w_e = -\left(\tau^x / f\rho_0\right)_y$$

- Cross-differentiation→

$$\beta hv = fw_e = -f\left(\frac{\tau^x}{f\rho_0}\right)_y$$

$$hh_x = -\frac{f^2}{g'\rho_0\beta}\left(\frac{\tau^x}{f}\right)_y \longrightarrow h^2 = h_e^2 + \frac{2f^2}{g'\rho_0\beta}\left(\frac{\tau^x}{f}\right)_y (x_e - x)$$

# The interior solution, based on Ekman pumping

- Integrating the vorticity equation leads to the meridional transport (the Sverdrup function)

$$m = \frac{f}{\rho_0 \beta}\left(\frac{\tau^x}{f}\right)_y (x_e - x)$$

There is no streamfunction because there is a mass source

- Introduce a virtual streamfunction

$$\psi^* = \frac{g'}{2 f_0}(h^2 - h_e^2) = \frac{f^2}{f_0 \rho_0 \beta}\left(\frac{\tau^x}{f}\right)_y (x_e - x)$$

- A constant line of $\psi^*$ is a streamline; but the flux between two lines can change due to Ekman pumping ⟶

The application to interior communication window later.

# Scaling of momentum equation in the western boundary layer

$$Assume\ scales: u \sim 0.01, v \sim 1, dx \sim 10^4, dy \sim 10^6,$$

$$\tau \sim 0.1, h \sim 100, g' \sim 0.01, R \sim 10^{-4}$$

$$hu_t + h(uu_x + vu_y) - fhv = -g'hh_x + \tau^x / \rho_0 + A\nabla_h{}^2(hu) - Ru$$

$$10^2(10^{-9}, 10^{-7}) \quad 10^{-2} \quad 10^{-2} \quad 10^{-4} \quad 10^{-6}$$

$$hv_t + h(uv_x + vv_y) + fhu = -g'hh_y + \tau^y / \rho_0 + A\nabla_h{}^2(hv) - Rv$$

$$10^2(10^{-6}, 10^{-6}) \quad 10^{-4} \quad 10^{-4} \quad 10^{-4} \quad ?? \quad 10^{-4}$$

Semi-geostrophy: geostrophy in the cross-stream direction
ageostrophy in the along-stream direction

$$fhv = g'hh_x$$

# Common features in Western boundary layer

- Integrating the geostrophic relation in cross-stream direction

$$fhv = g'hh_x$$

$$h_I^2 = h_e^2 + \frac{2f^2}{g'\rho_0\beta}\left(\frac{\tau^x}{f}\right)_y (x_e - x_w)$$

$$\psi = \psi_I + \frac{g'}{2f}\left(h^2 - h_I^2\right)$$

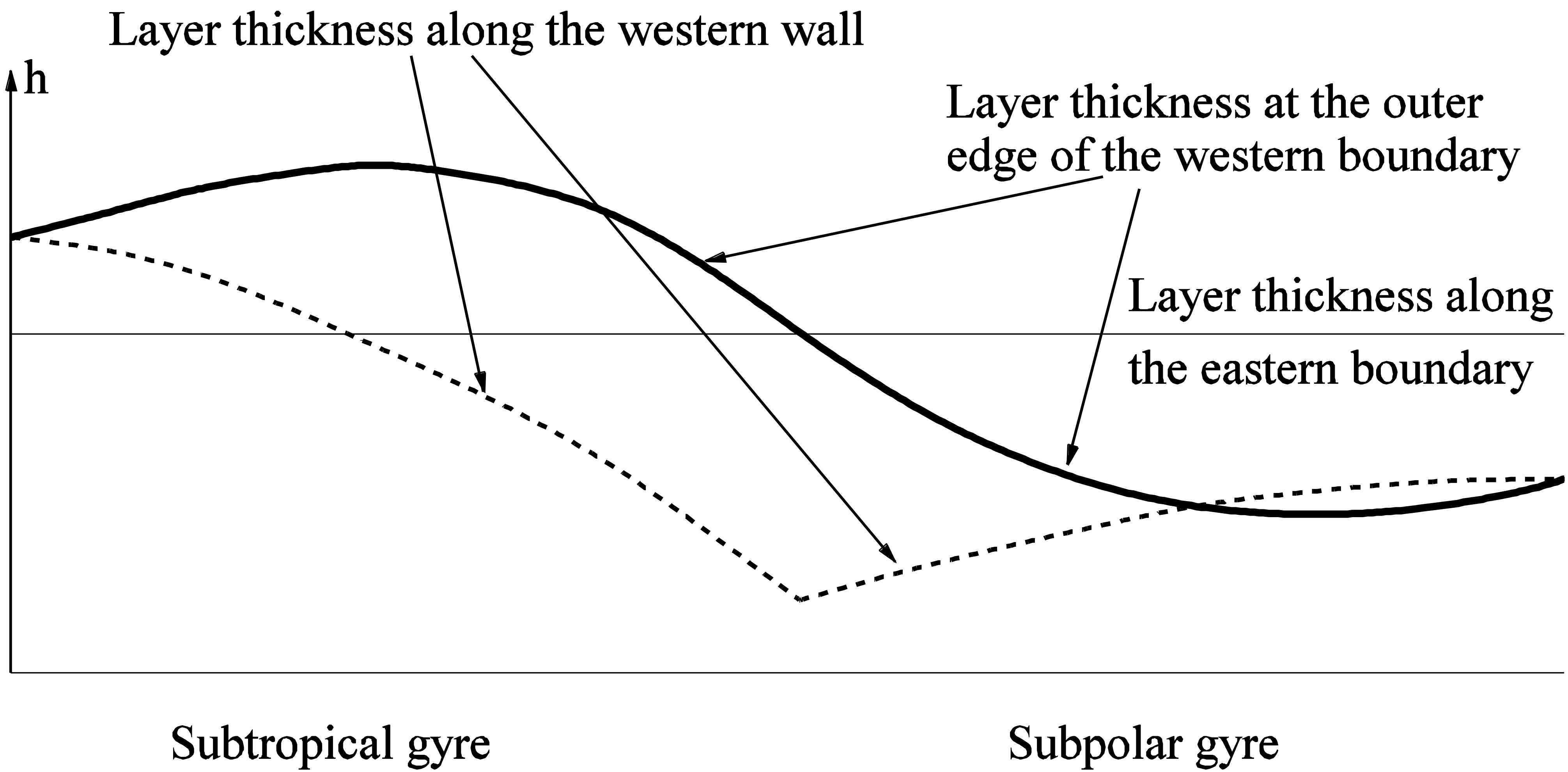
$$\psi_I = \frac{1}{\rho_0\beta}\tau_y^x (x_e - x_w)$$

$$h_w^2 = h_I^2 - \frac{2f}{g'}\psi_I = h_e^2 - \frac{2}{g'\rho_0}\tau^x\left(x_e - x_w\right)$$

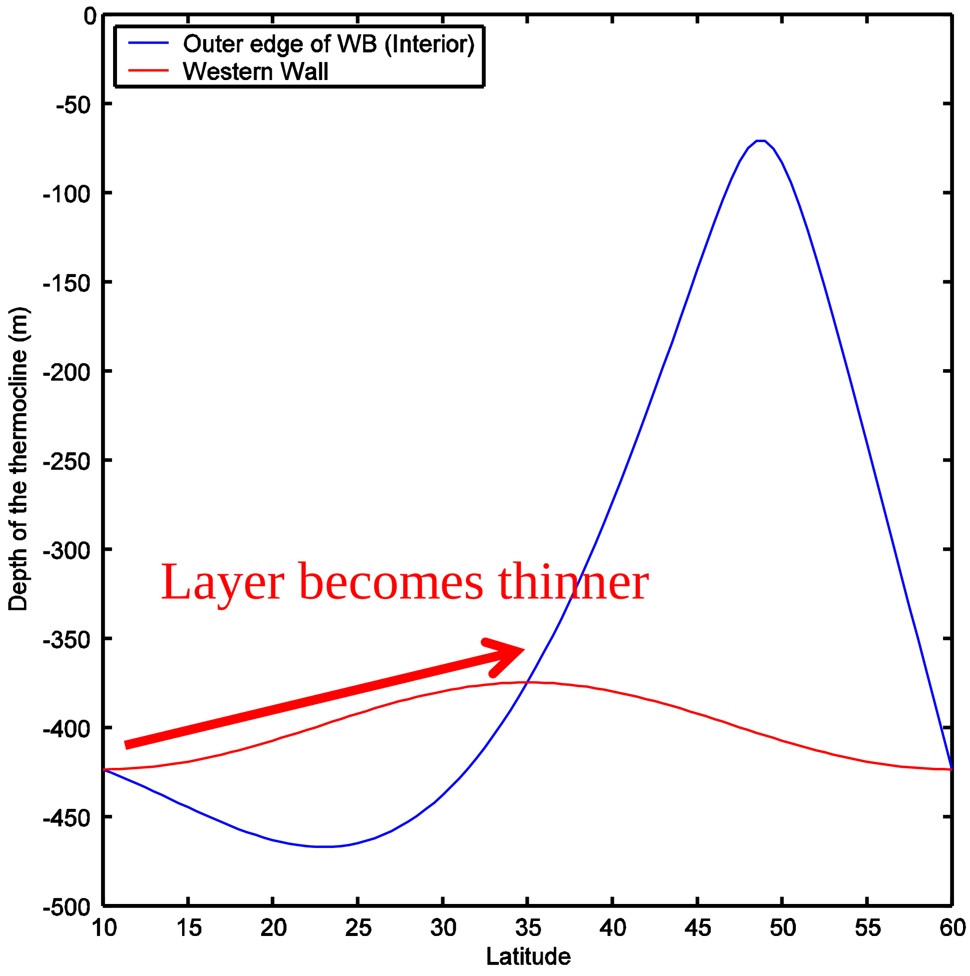
1) Large layer thickness change across stream

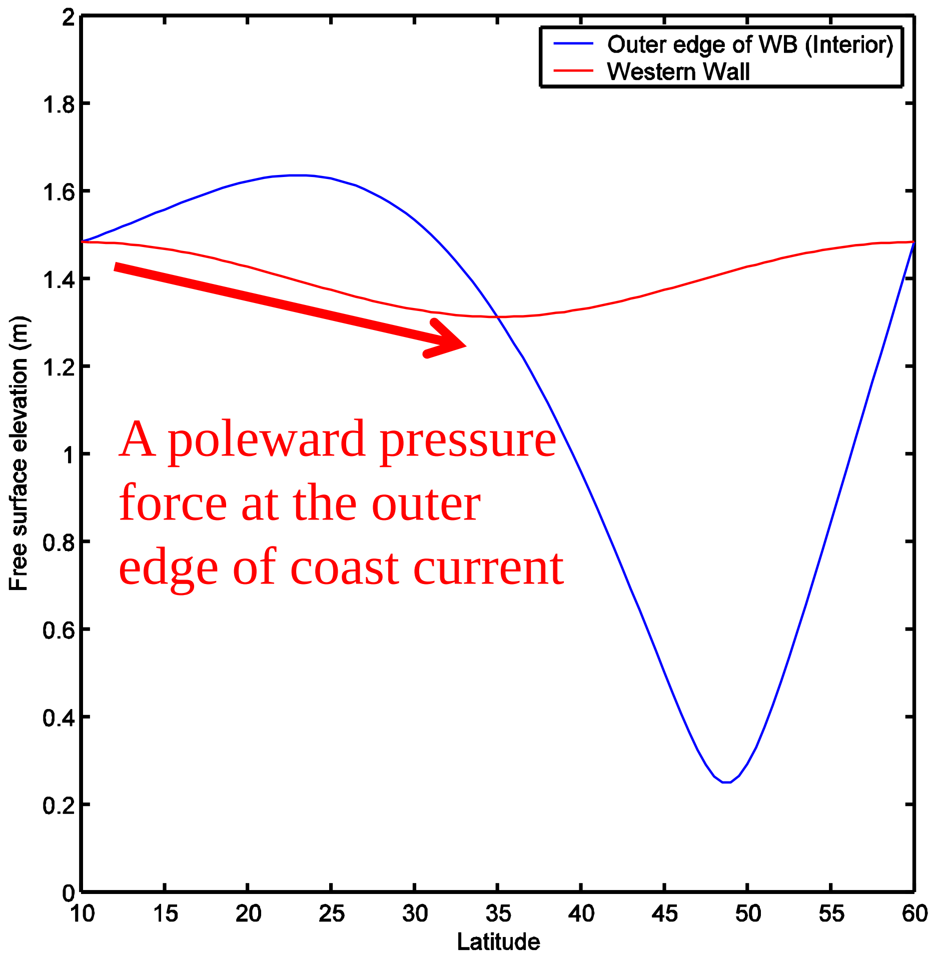
2) Within the subtropical gyre $\frac{\partial h_w^2}{\partial y} = -\frac{2}{g'\rho_0}\left(x_e - x_w\right)\frac{\partial \tau^x}{\partial y} < 0$

# Thermocline thickness at the outer edge and wall of the western boundary



# A numerical example of the thermocline thickness and surface elevation



这样一个北向的压强梯度力是沿岸环流(其在海岸横向的尺度则小得多)的一个重要的远岸背景场。一方面,在亚热带海盆,这个经向压强梯度力可以推动一支沿着亚热带海盆西边界的向极沿岸流。另一方面,在亚极地海盆,对应的压强梯度是向赤道的,因而它在建立沿着亚极带海盆西边界的向赤道流动中发挥了关键的作用。此外,由于在整个海盆上气候条件的变化,这样一个大尺度的压强场应该在年代的时间尺度上发生变化,由此而带来沿岸环流的变化。

# Scaling in the western boundary layer

$$hu_t + h(uu_x + vu_y) - fhv = -g'hh_x + \tau^x/\rho_0 + A\nabla_h^{\ 2}(hu) - Ru$$

$$10^2(10^{-9}, 10^{-7}) \quad 10^{-2} \quad 10^{-2} \quad 10^{-4} \quad 10^{-6}$$

$$hv_t + h(uv_x + vv_y) + fhu = -g'hh_y + \tau^y/\rho_0 + A\nabla_h^{\ 2}(hv) - Rv$$

$$10^2(10^{-6}, 10^{-6}) \quad 10^{-4} \quad 10^{-4} \quad 10^{-4} \quad ?? \quad 10^{-4}$$

$$-fhv = -g'hh_x$$

$$fhu = -g'hh_y - Rv$$

$$\longrightarrow \quad \beta hv = -Rv_x$$

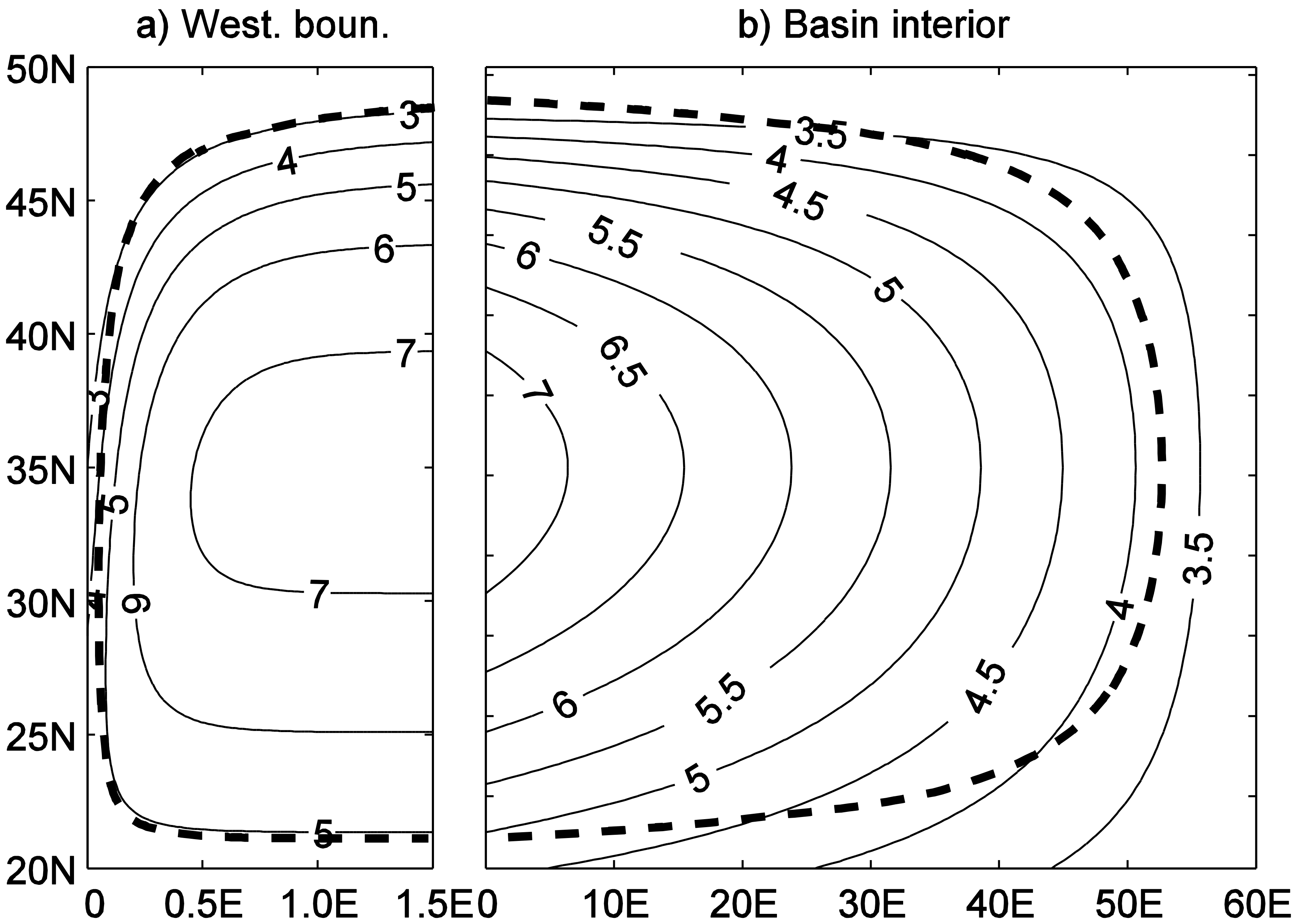
$$\beta hv = -\frac{\tau^x_y}{\rho_0} + R\left(u_y - v_x\right)$$

$$2\cdot 10^{-9} \quad 10^{-10} \quad 2\cdot 10^{-4}\left(10^{-8}, 10^{-5}\right)$$

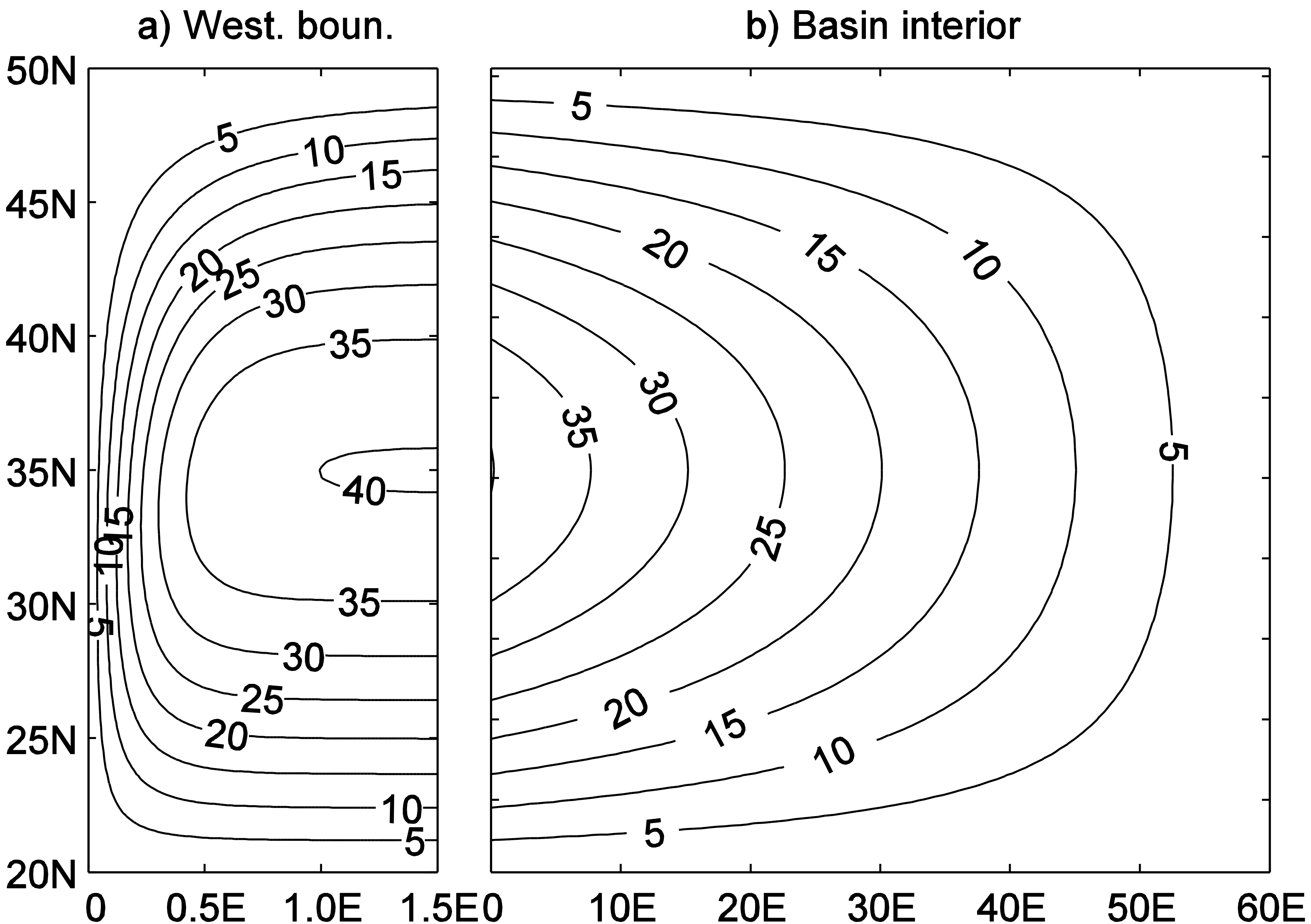
Scaling of the more complete PV equation→ wind-stress curl and $Ru_y$ can be neglected

# Thermocline depth with a Stommel layer

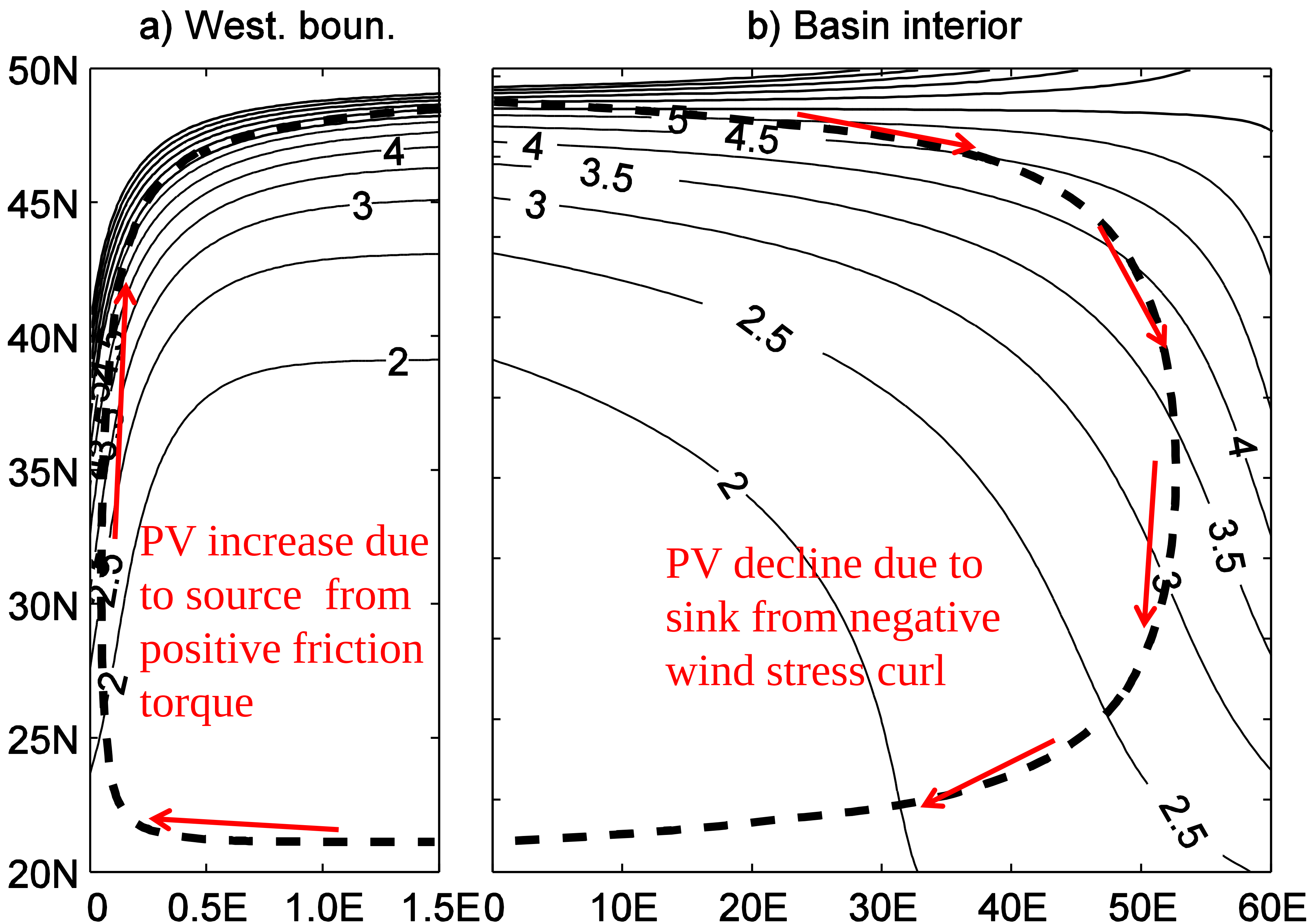
$$\tau^x = -0.15\cos\left(\frac{y-y_s}{y_n-y_s}\pi\right), g' = 0.015m/s^2, h_e = 300m$$



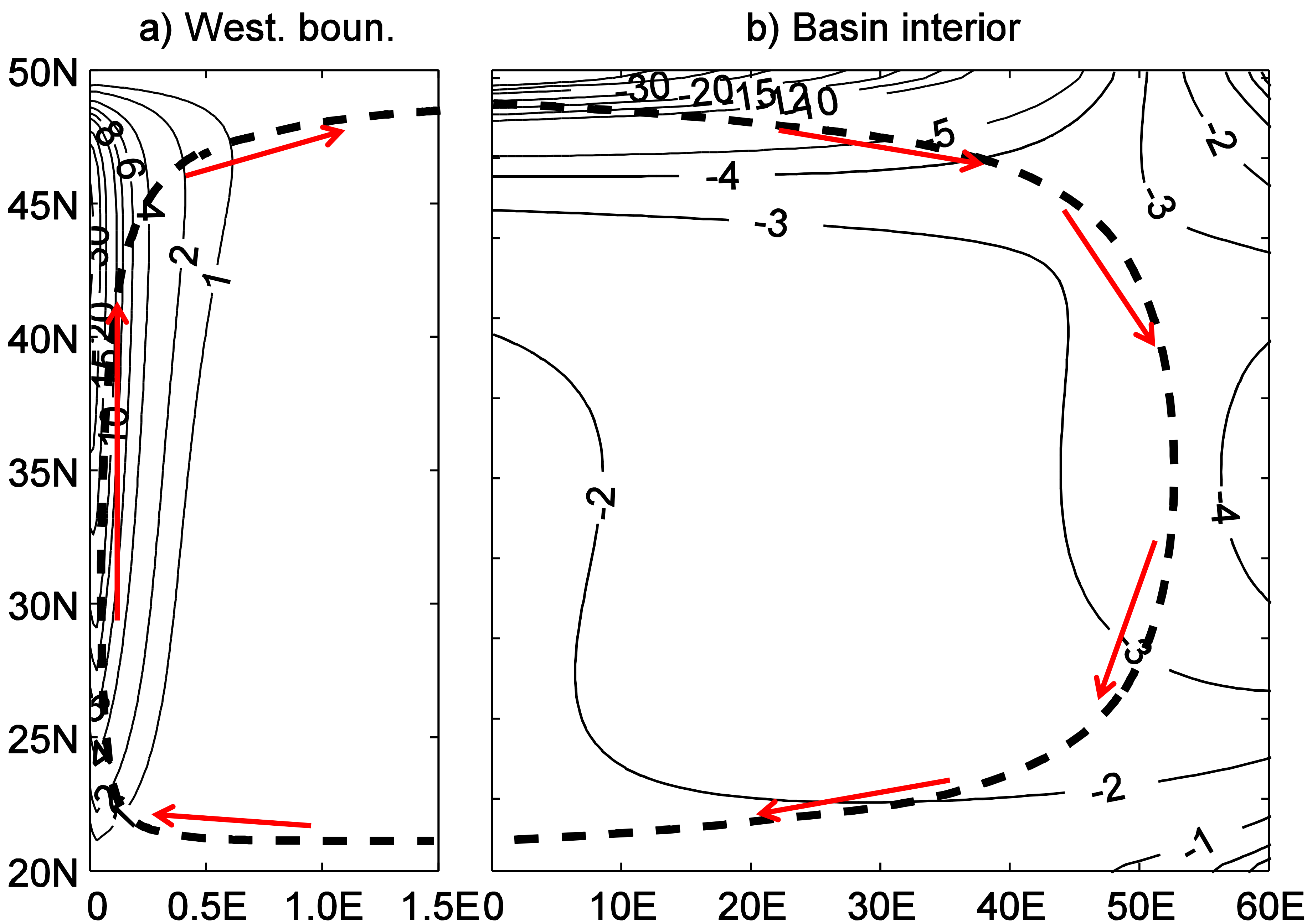
# Wind-driven gyre with a Stommel layer (in Sv)



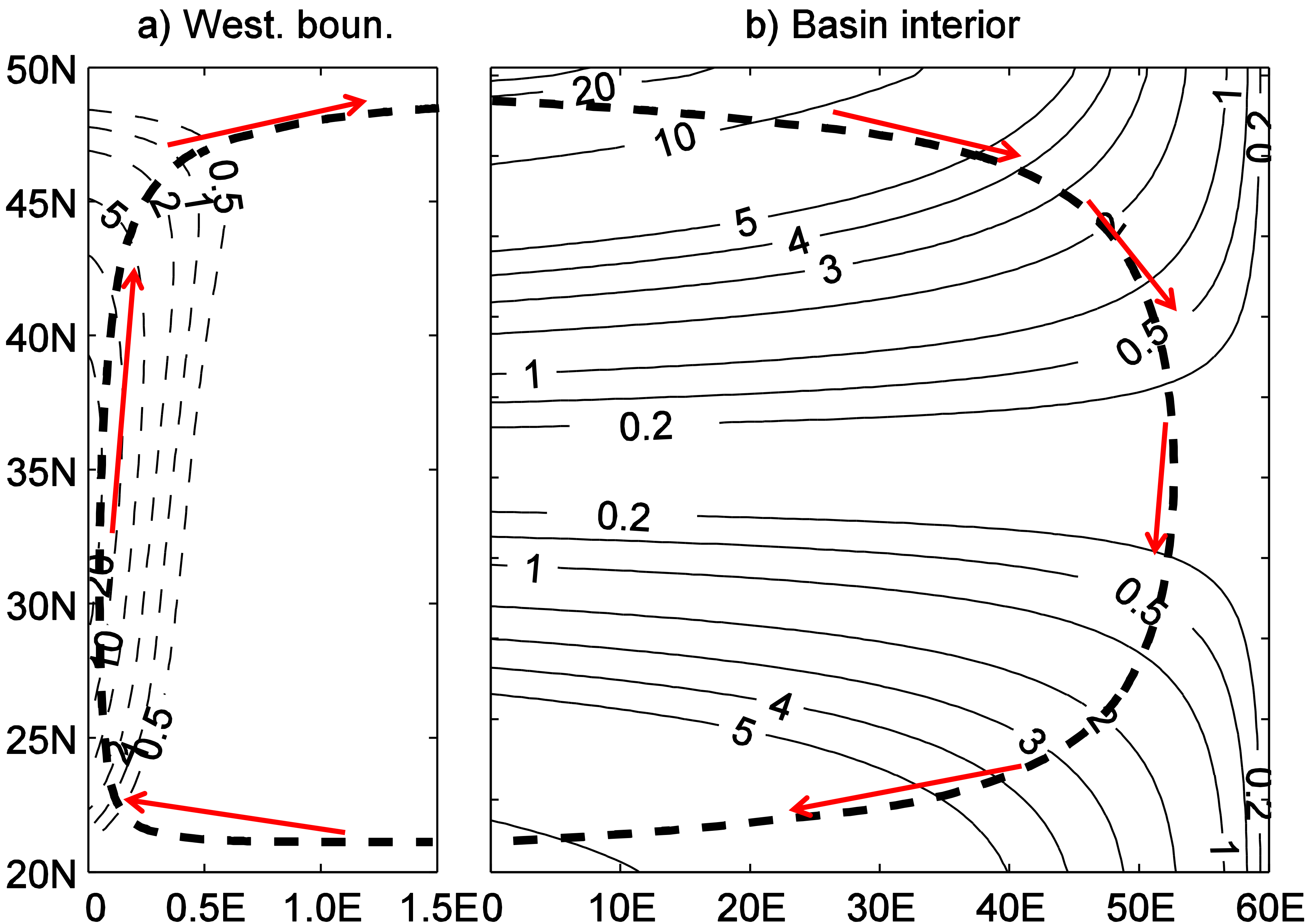
# Potential vorticity ($10^{-10}$/s/m)



# The sink and source of PV



# Wind work on the subtropical gyre



# The Inertial western boundary layer

- Basic idea → Stommel (1954): within the western boundary, the inertial term associated with horizontal advection balances the planetary vorticity term

  Charney (1955) and Morgan (1956) put this idea in accurate formulation

- The basic equations

$$h(uu_x + vu_y) - fhv = -g'hh_x + \tau^x / \rho_0$$

$$h(uv_x + vv_y) + fhu = -g'hh_y$$

$$(hu)_x + (hv)_y = 0$$

$$fhv = g'hh_x$$

$$B = \frac{1}{2}v^2 + g'h = F(\psi), \quad \textit{energy conservation}$$

$$Q = \frac{f + v_x}{h} = G(\psi), \quad \textit{potential vorticity conservation}$$

# The inertial western boundary layer

- Determine the energy/vorticity functions from the outer edge of the w.b.l.

$$F(\psi_I) = g' h_I(Y) \qquad G(\psi_I) = \frac{f(Y)}{h_I(Y)}$$

- The interior solution at the outer edge of w.b.l.

$$\psi_I = \frac{x_e}{\rho_0 \beta} \tau_y^x(Y)$$

Define the inverse function

$$Y = Y(\psi_I)$$

- The final forms

$$F(\psi_I) = g' h_I\left[Y\left(\psi_I\right)\right] \qquad G(\psi_I) = \frac{f\left[Y\left(\psi_I\right)\right]}{h_I\left[Y\left(\psi_I\right)\right]}$$

- North of the latitude where $\psi_I(Y)$ reaches the maximum, this one-to-one inverse is no longer valid → this is the northern limit of inertial western boundary layer

# The inertial western boundary layer

- Streamfunction and layer thickness within w.b.l.

$$\psi = \psi_I + \frac{g'}{2f}\left(h^2 - h_I^2\right) \qquad h^2 = h_I^2 + \frac{2f}{g'}\left(\psi - \psi_I\right)$$

- The velocity is linked through the Bernoulli function

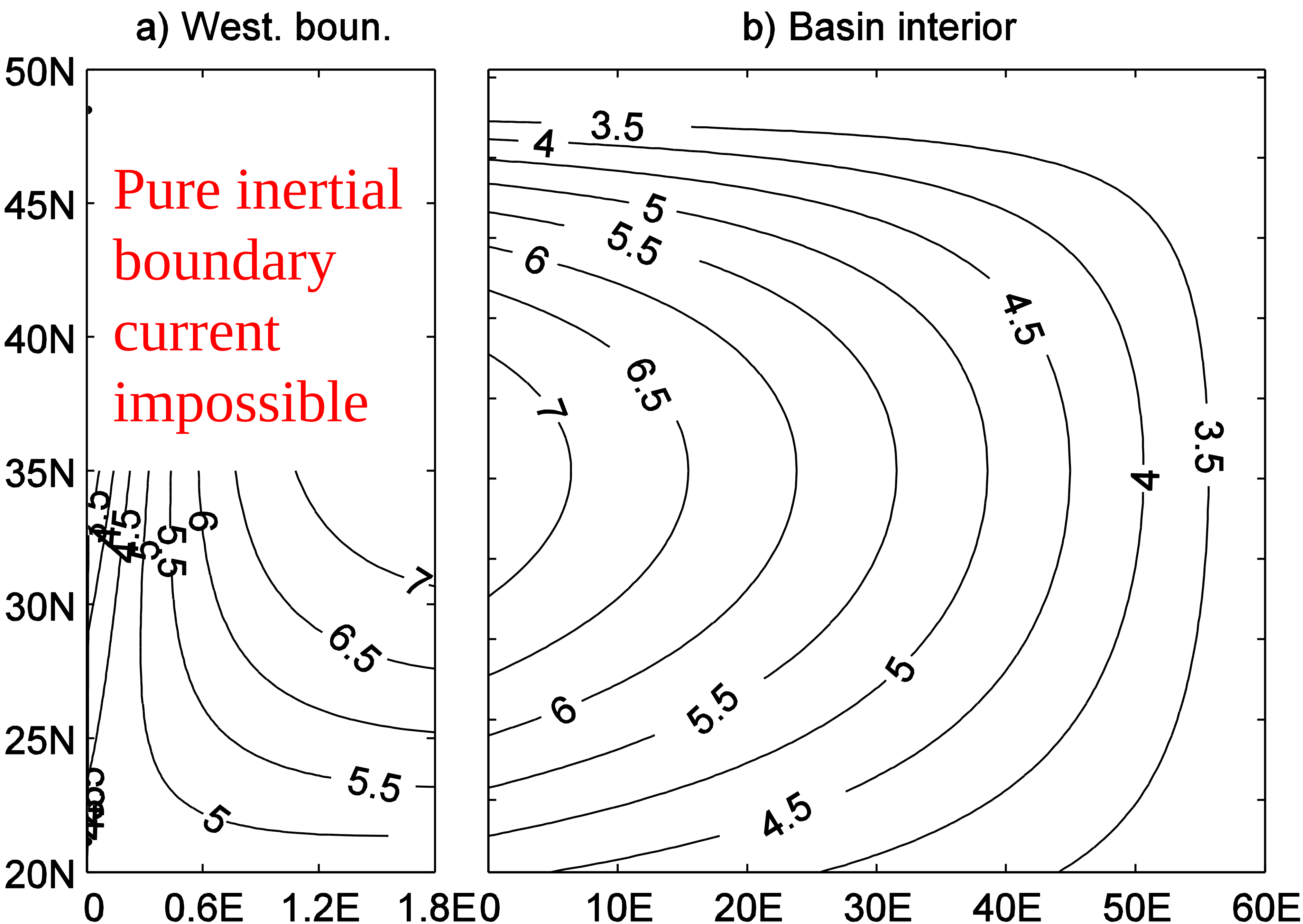
$$\frac{1}{2}v^2 + g'h = F(\psi) \Rightarrow v = \sqrt{2\left[F(\psi) - g'h\right]}$$

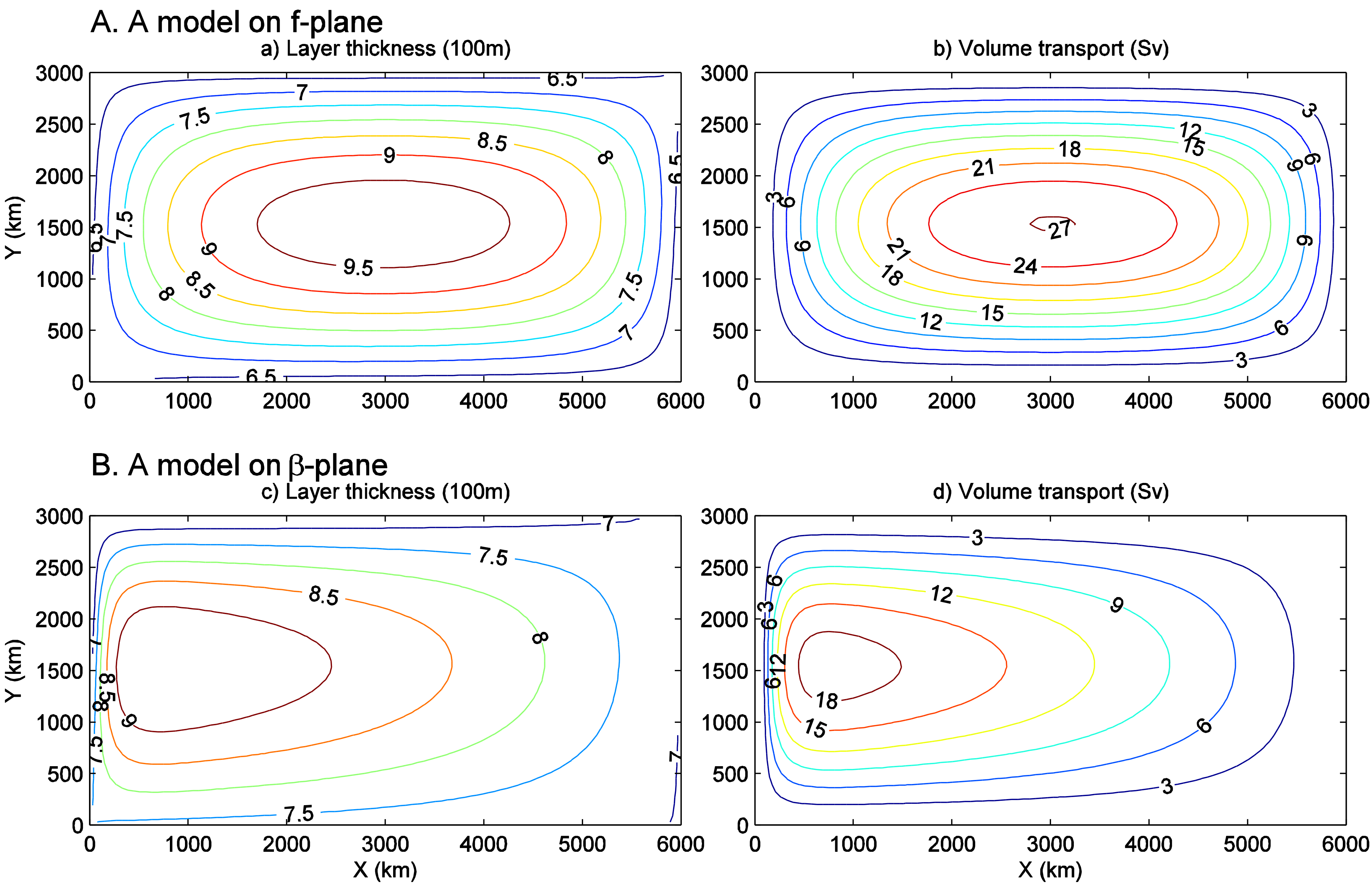
- Note that $h^2 = h_I^2 + \frac{2f}{g'}\left(\psi - \psi_I\right)$

is the inertial western boundary layer solution in streamfunction coordinate, which can be plot in physical coordinate by use the Von-Mises transformation (used in fluid mechanics) to the streamfunction coordinate

$$x = \int_0^{\psi} \frac{d\psi}{hv}$$

# Inertial western boundary current




A. A model on f-plane
a) Layer thickness (100m)
b) Volume transport (Sv)
B. A model on β-plane
c) Layer thickness (100m)
d) Volume transport (Sv)
Y (km)
X (km)
0
500
1000
1500
2000
2500
3000
1000
2000
3000
4000
5000
6000
6.5
7
7.5
8
8.5
9
9.5
3
6
9
12
15
18
21
24
27

# PV controls the lowest-order dynamics

## Other parameters (friction) are secondary

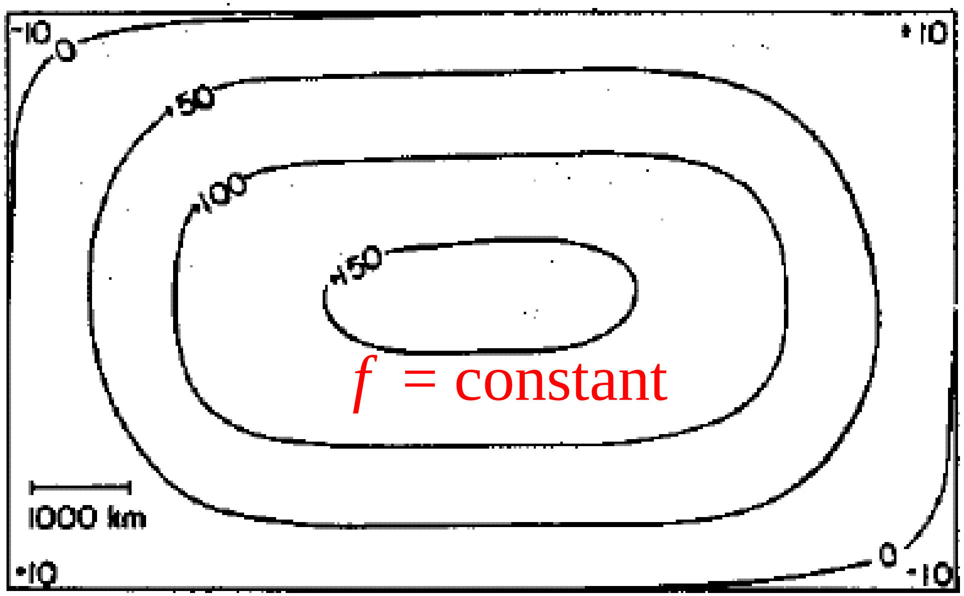



Fig. 4--Surface height contours for the uniformly rotating ocean in cm referred to an arbitrary level

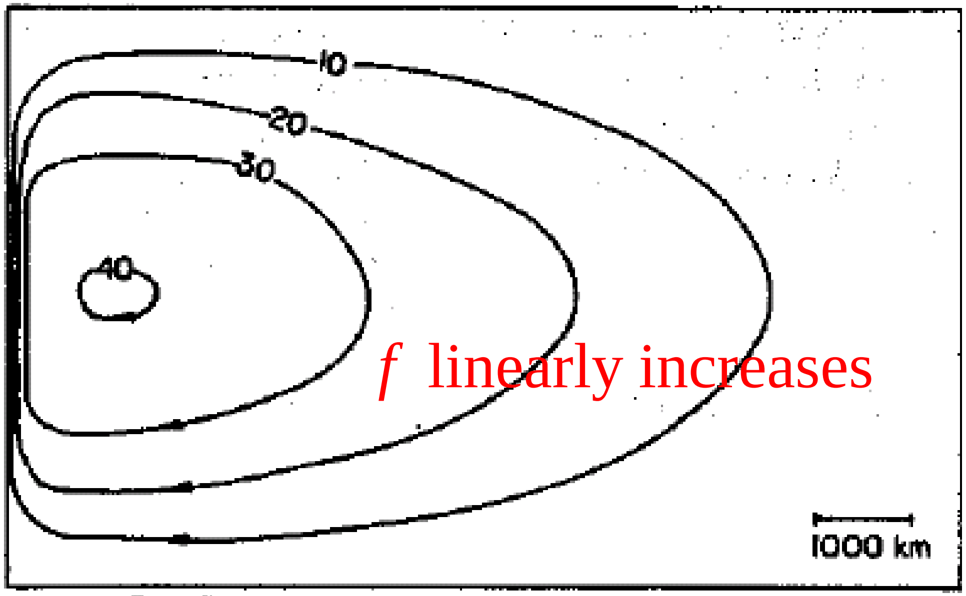



Fig. 5--Streamlines for the case where the Coriolis force is a linear function of latitude

Stommel (1948) classical figures

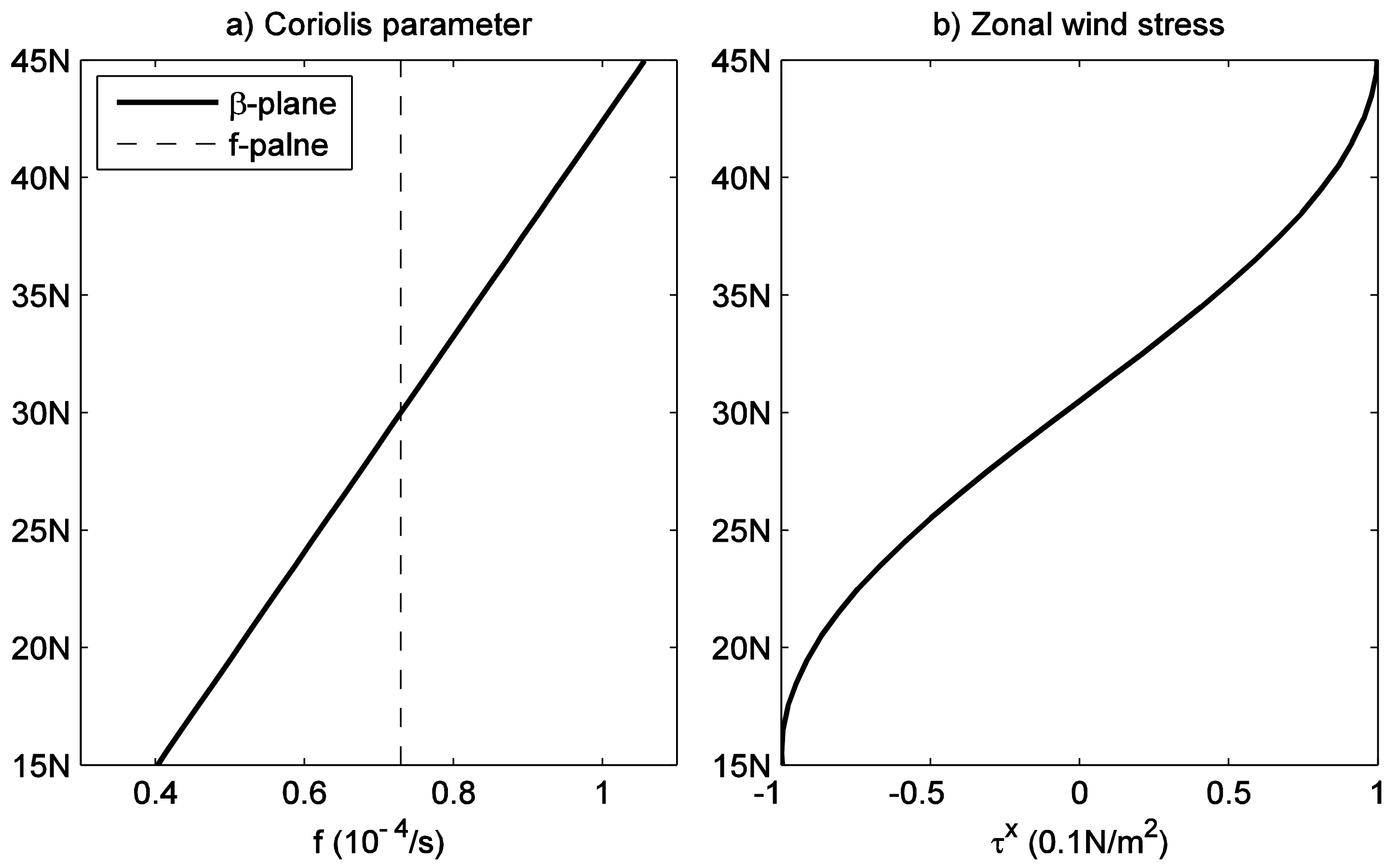
What controls the circulation?

# A simple model

$$hu_t - fhv = -g'hh_x + \tau^x/\rho_0 + A\nabla_h^2(hu) - ru$$

$$hv_t + fhu = -g'hh_y + \tau^y/\rho_0 + A\nabla_h^2(hv) - rv$$

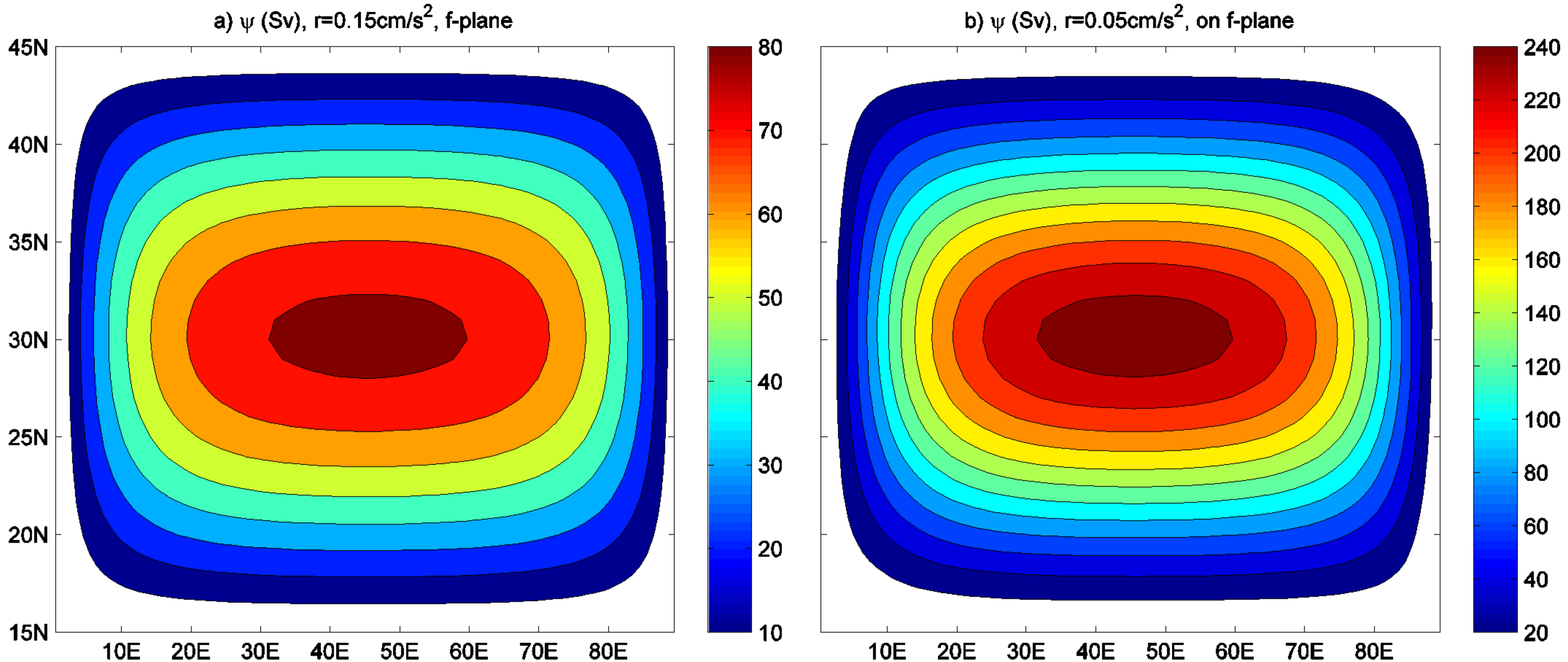
$$h_t + (hu)_x + (hv)_y = 0$$



# Streamfunction for a model on f-plane
## Circulation inversely proportional to friction

Vorticity equation

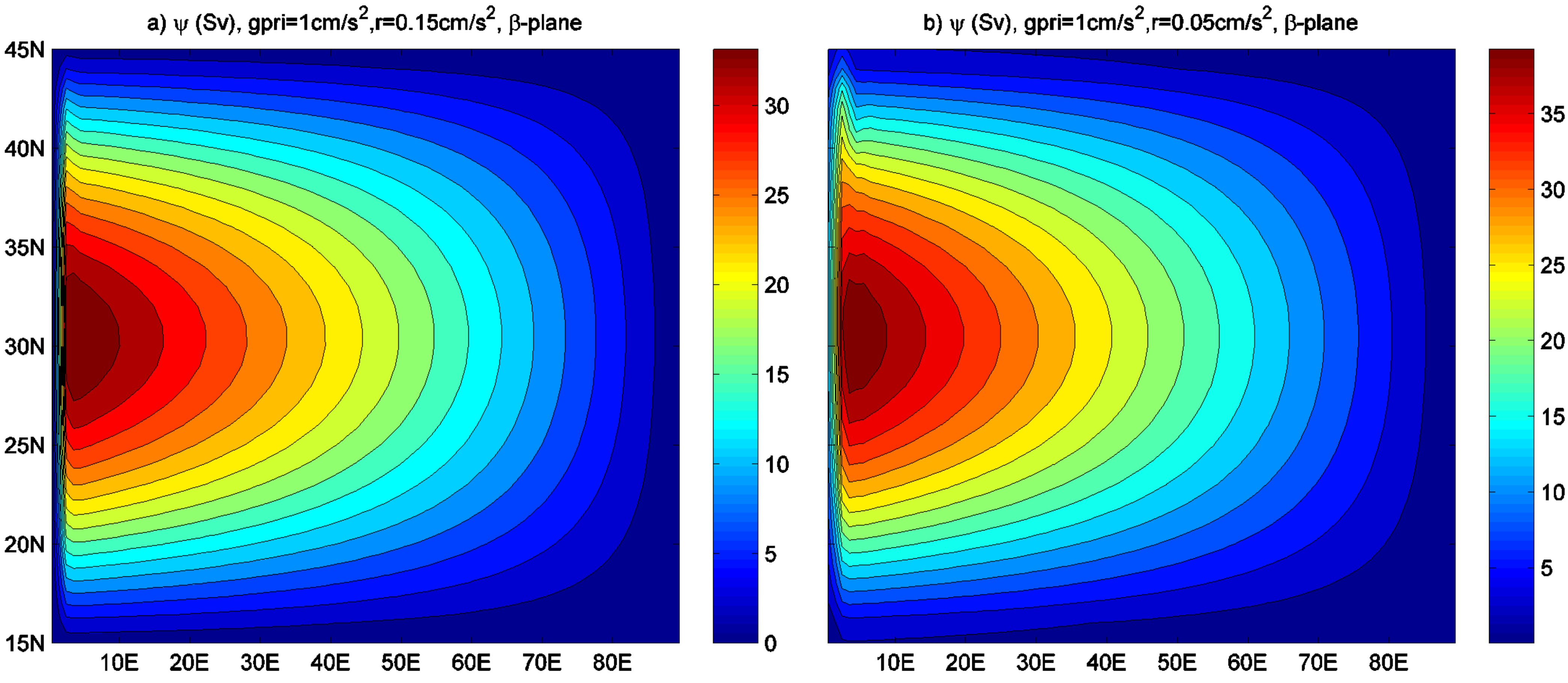
$$\left(\tau^{x}\right)_{y} / \rho_{0} = r\left(u_{y} - v_{x}\right) + HOT$$



# Streamfunction for a model on beta-plane solution insensitive to friction

Vorticity equation $\beta h v = -\tau_y^x / \rho_0 + r\left(u_y - v_x\right)$

High order term
Necessary only in the western boundary

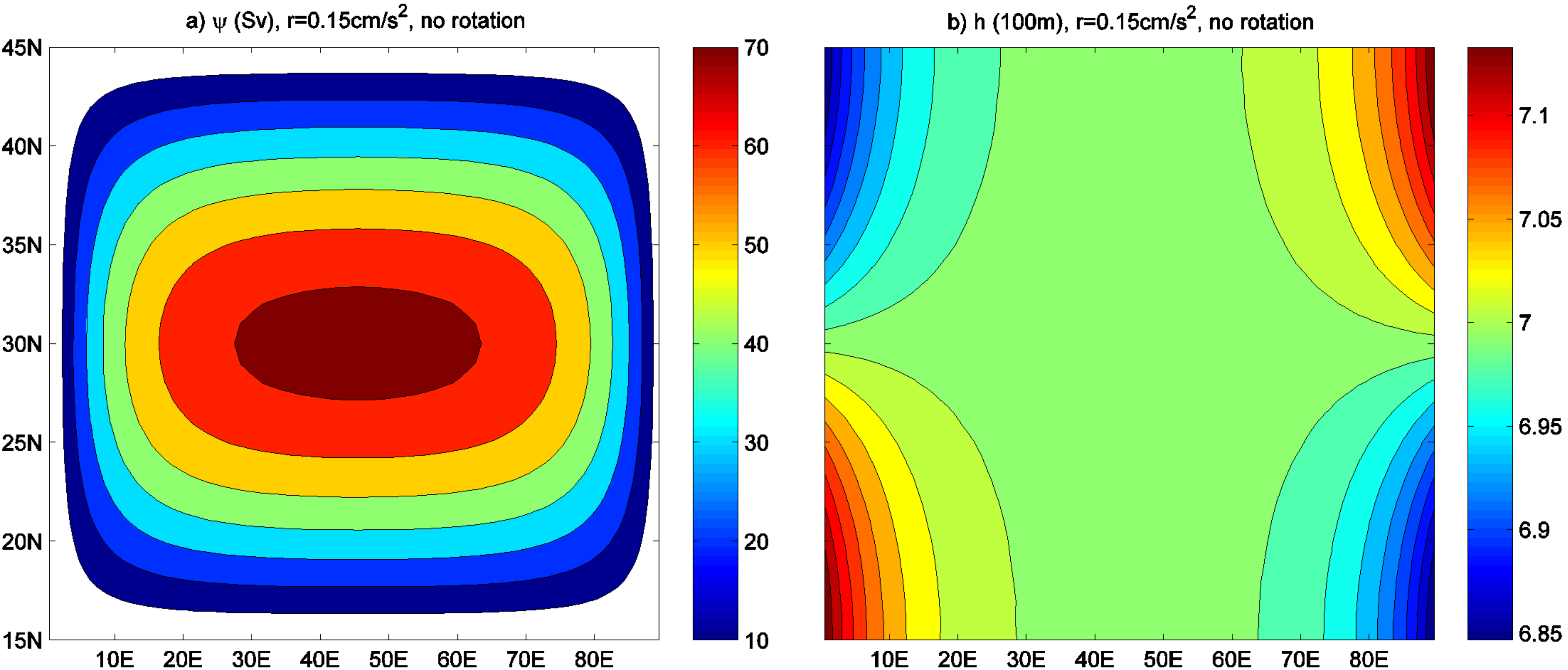


# A model with no rotation

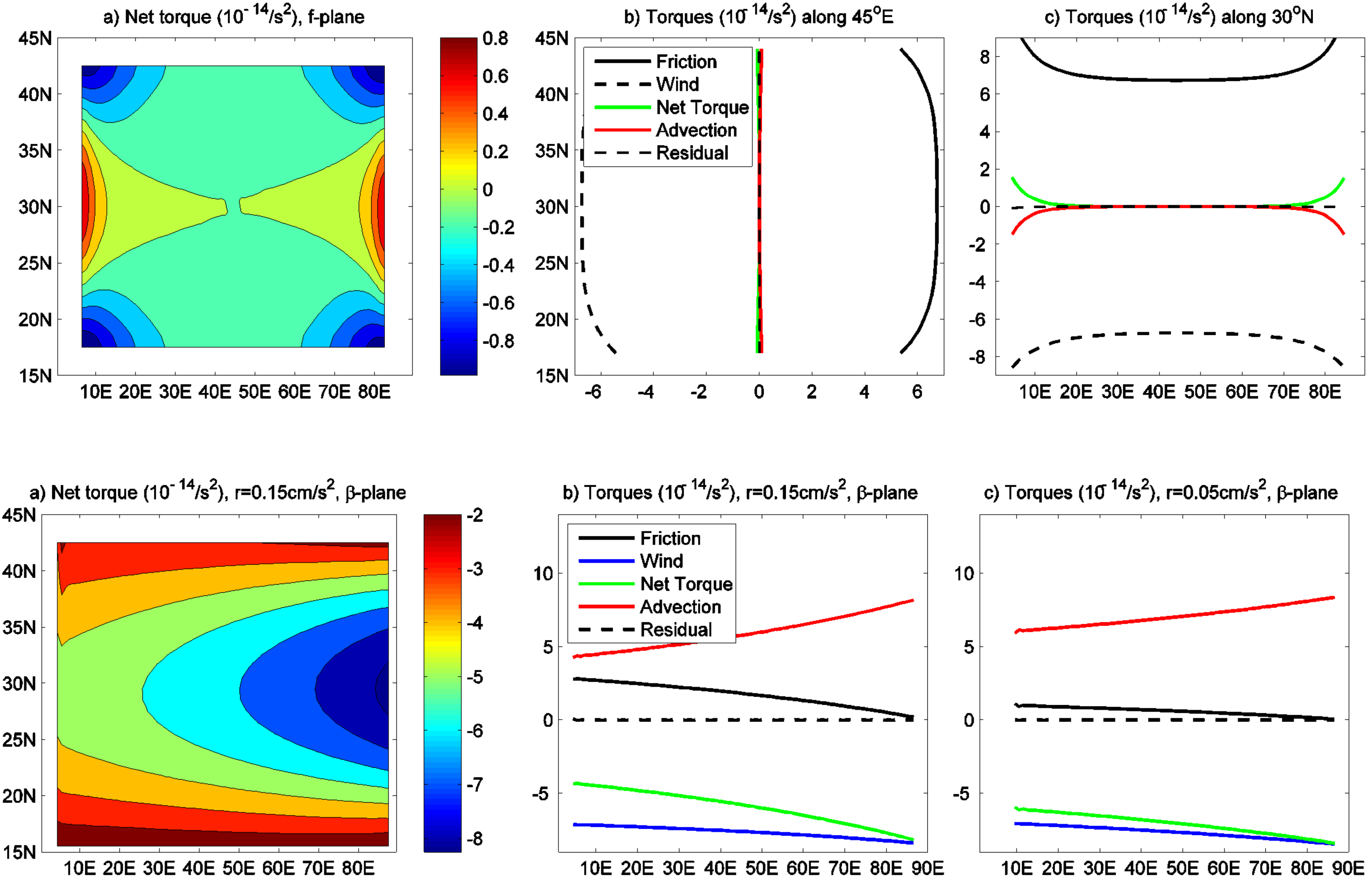
## Pressure gradient is nearly zero

Vorticity equation

$$\left(\tau^x\right)_y / \rho_0 = r\left(u_y - v_x\right) + HOT$$

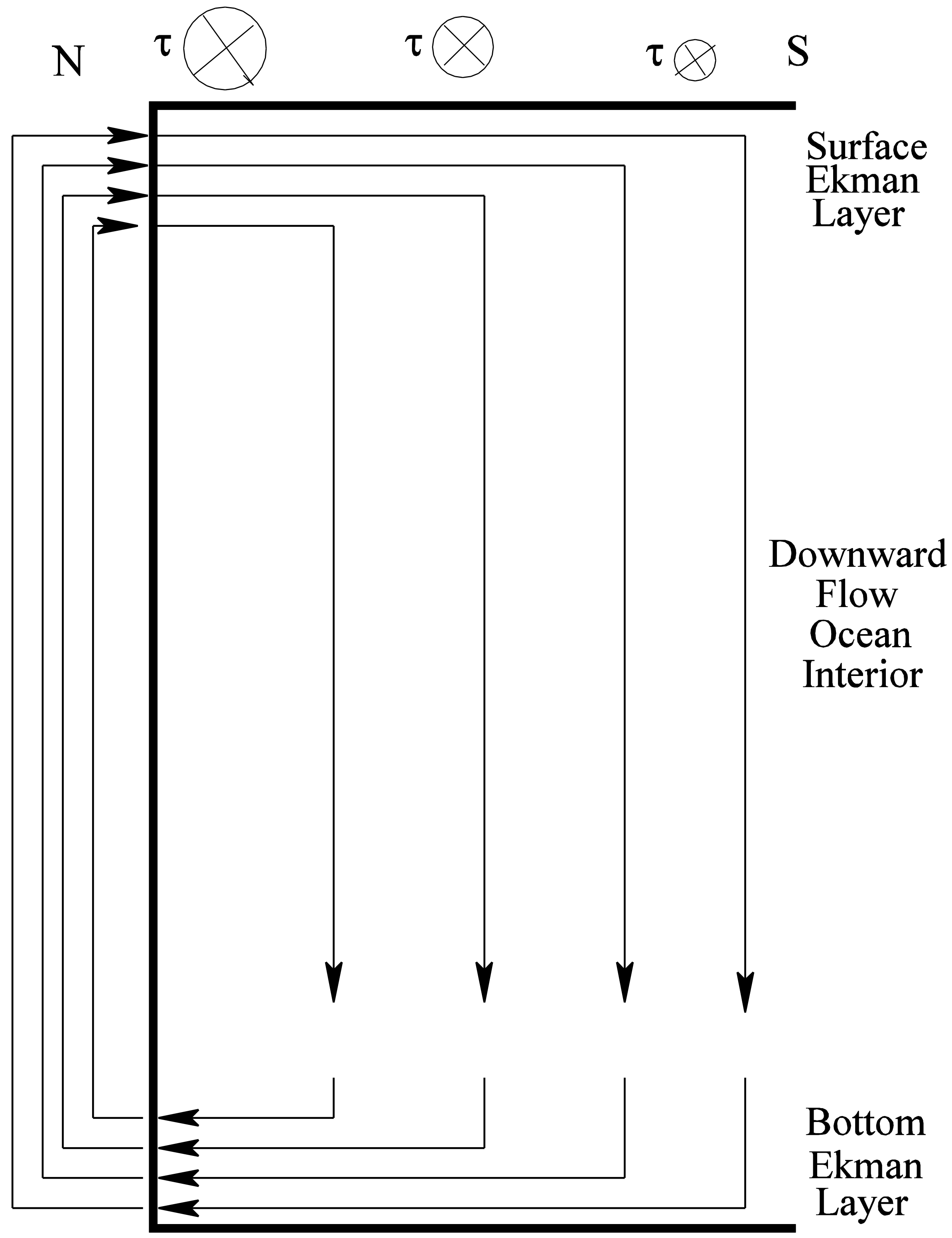


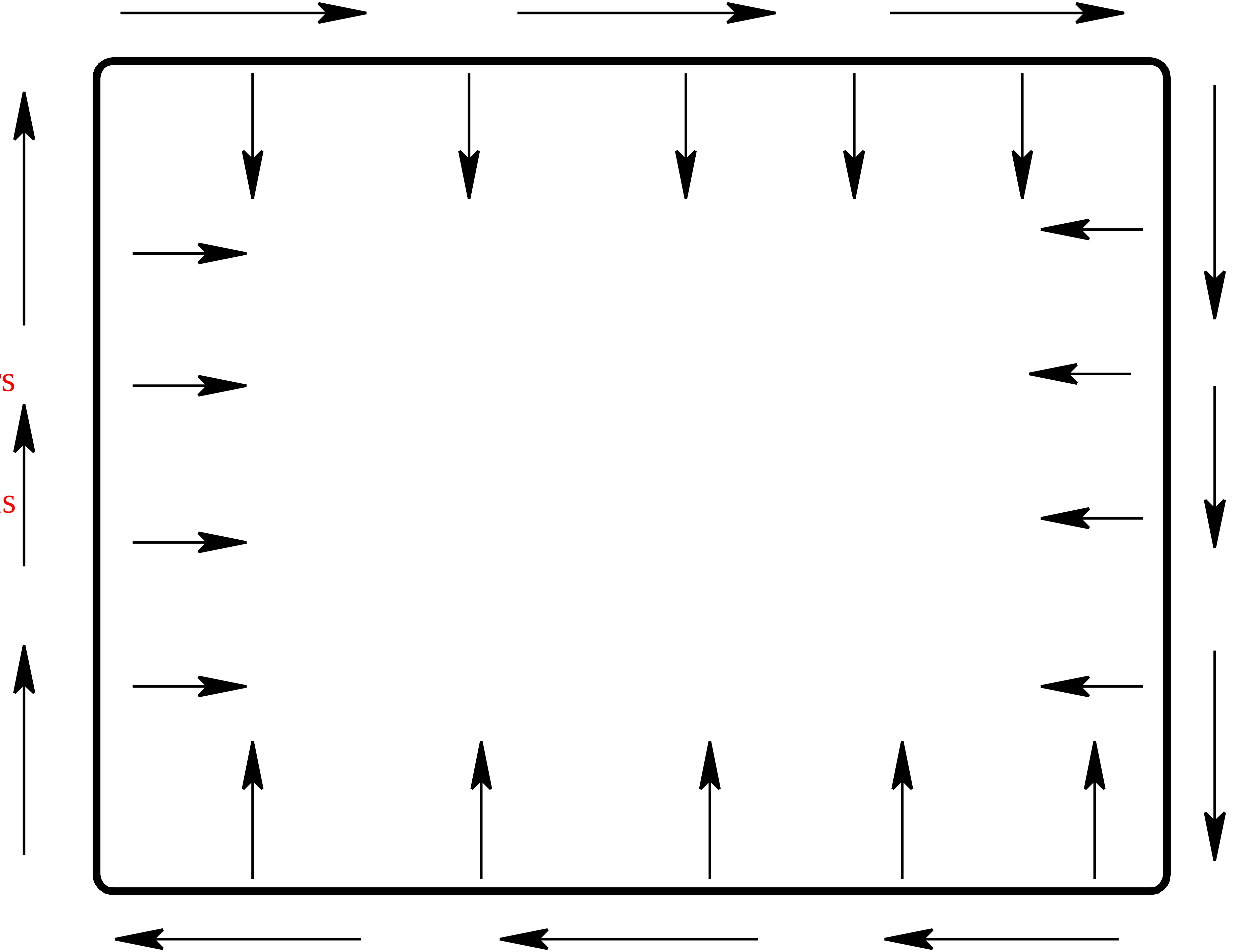
# Vorticity balance



Some thought about Ekman cells

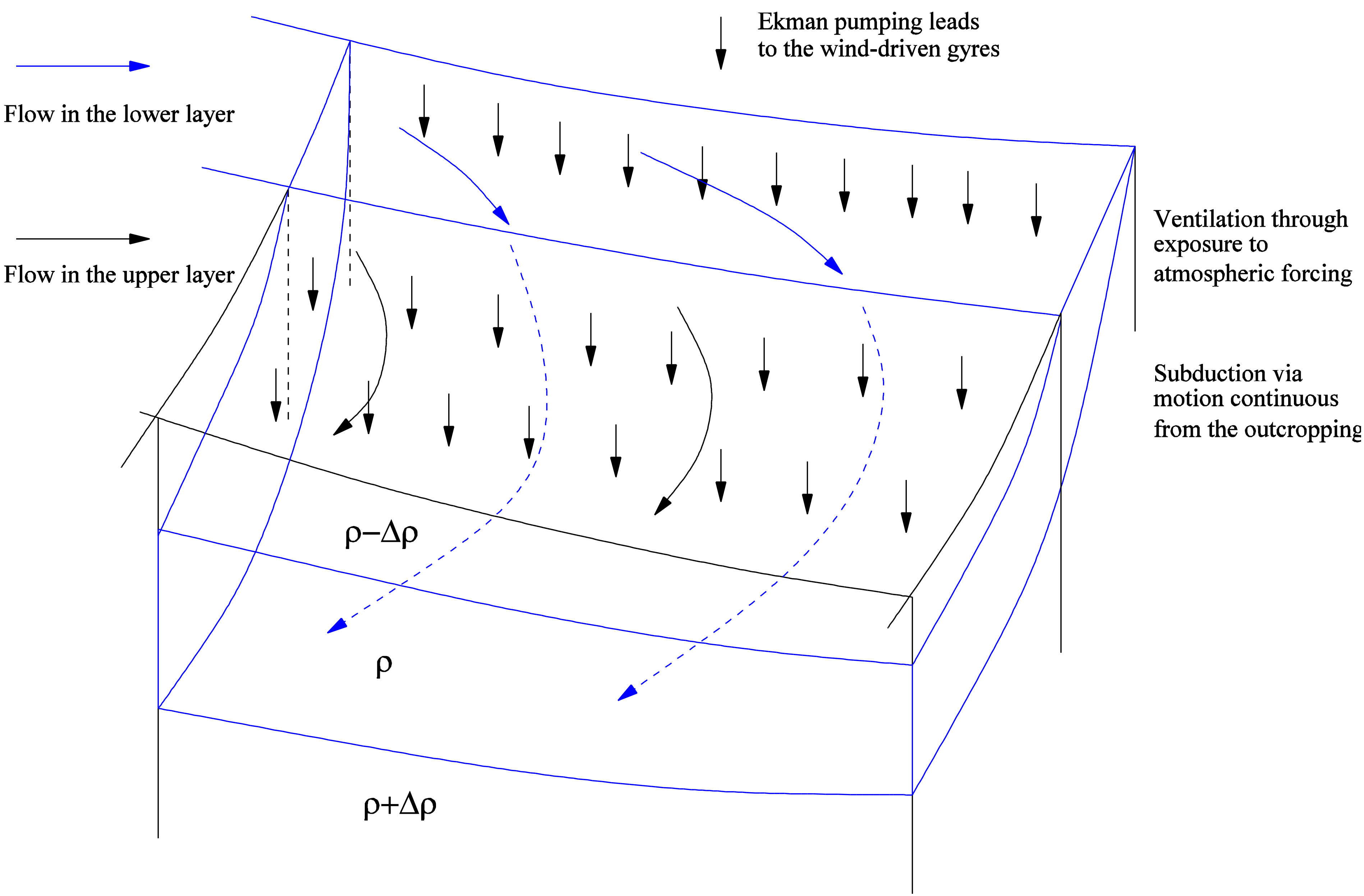
Two Ekman Layers for a Homogeneous fluid



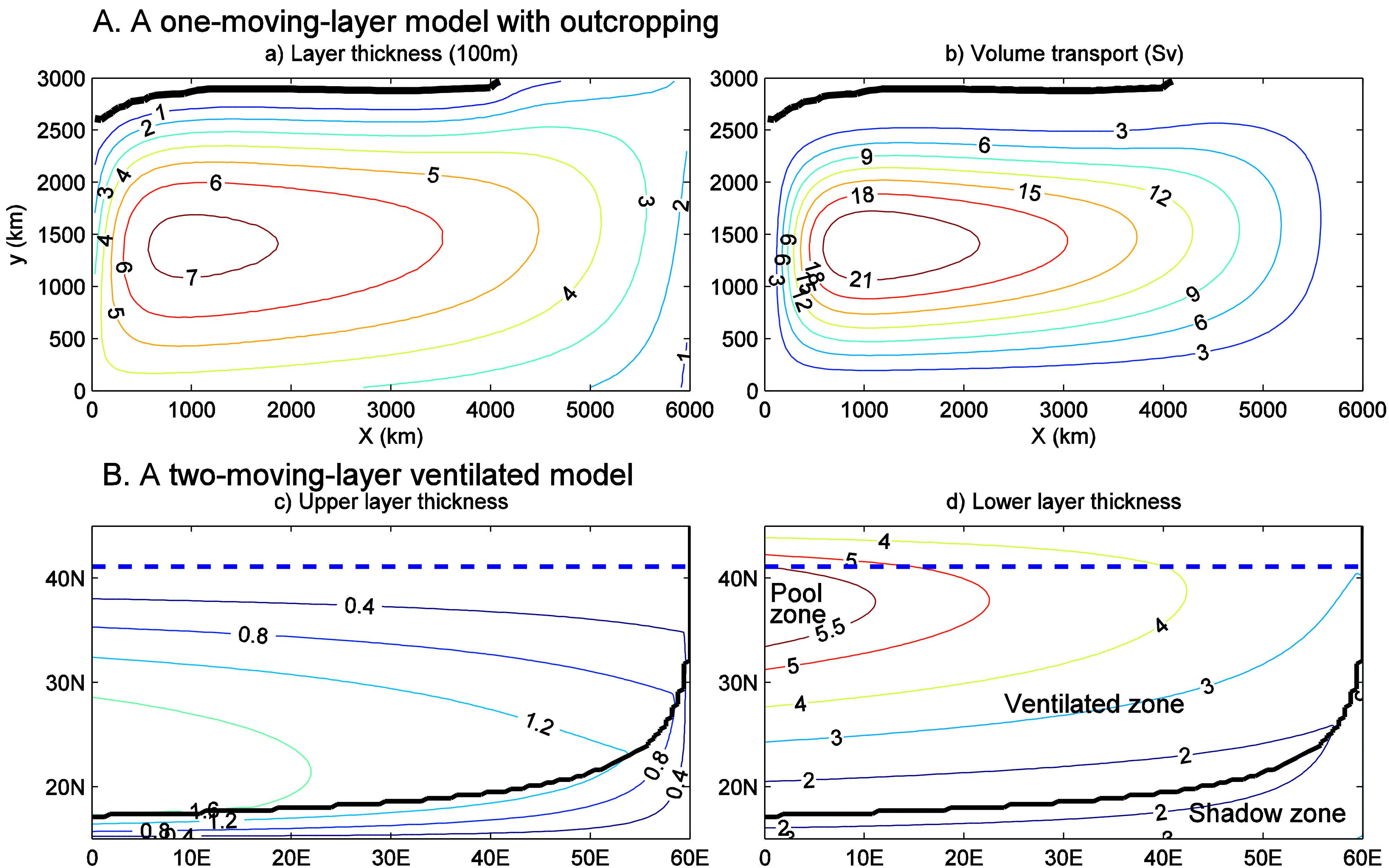


Ekman layers in a square basin with homegeneous fluid

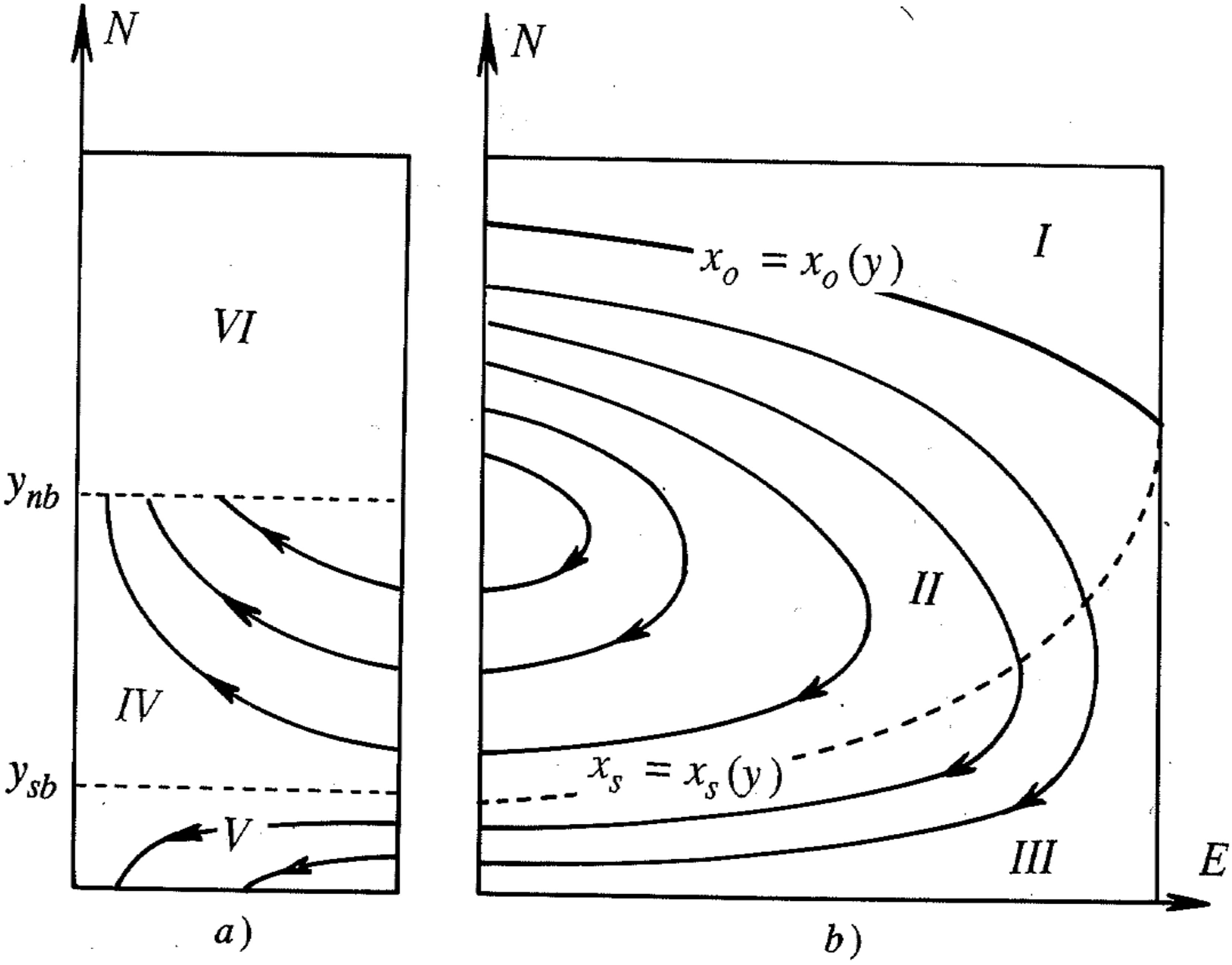
# Ventilation and subduction



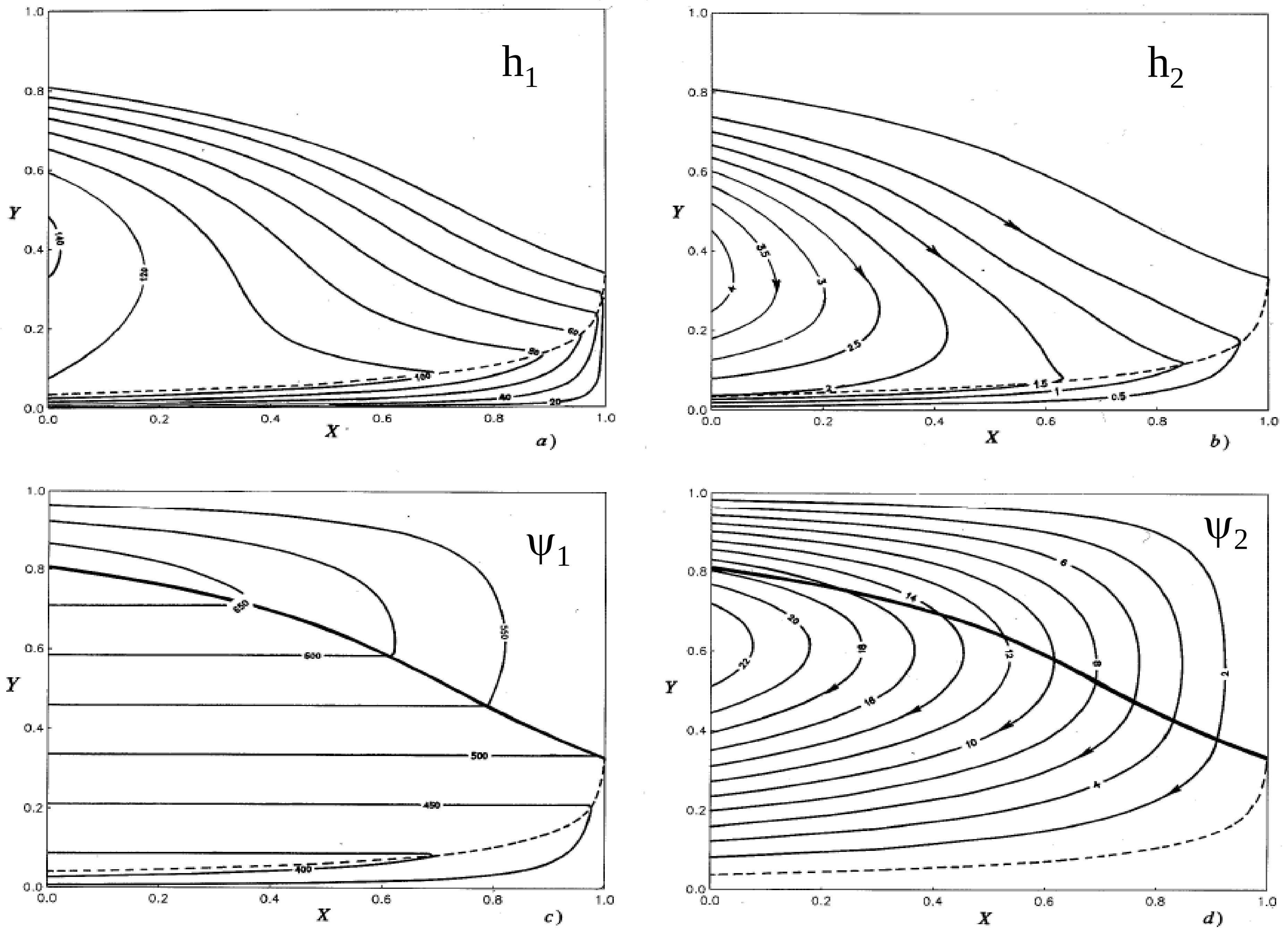
# Ventilated thermocline



# A 2-moving layer ventilated thermocline including IWBC

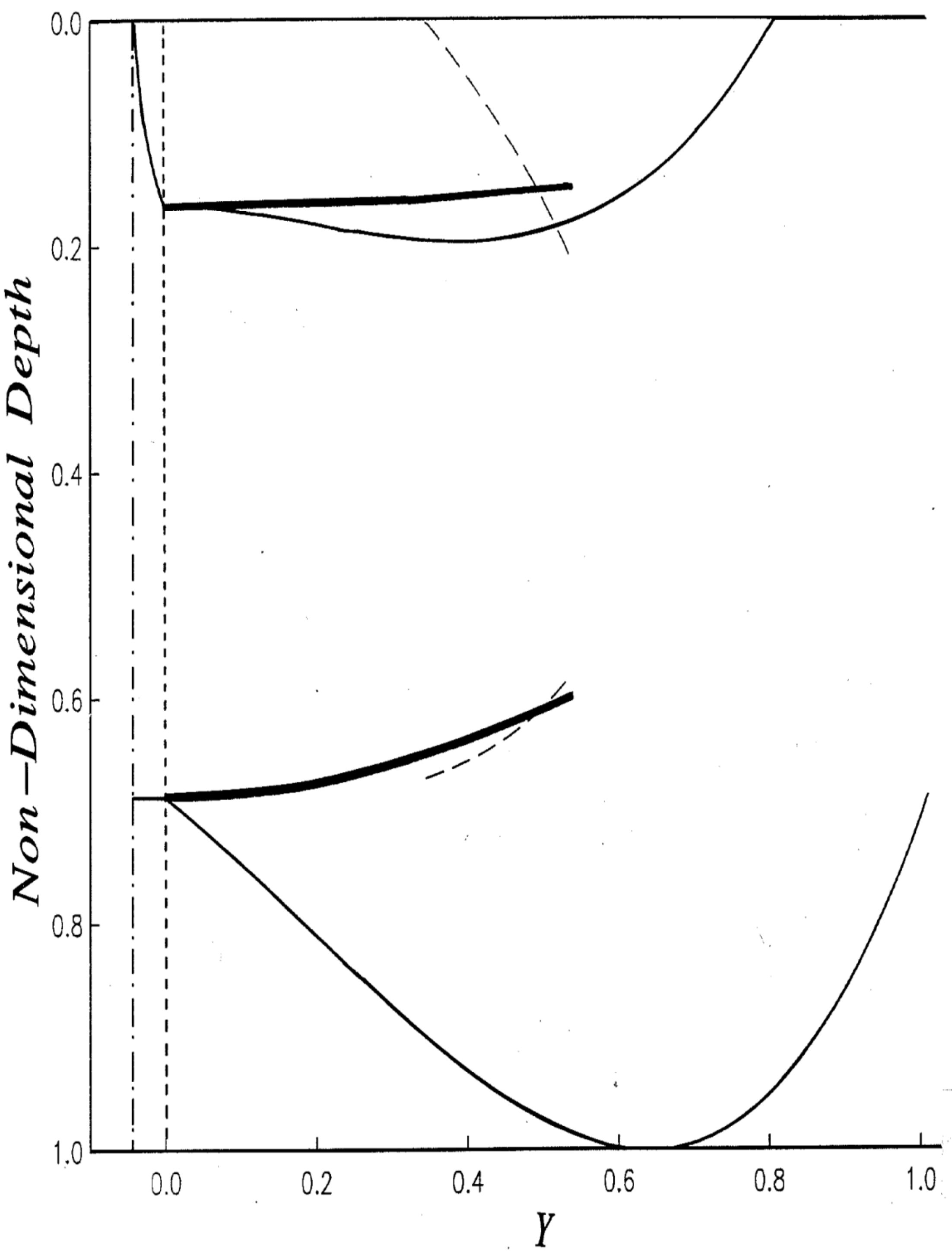


# Interior solution (2 moving layers)

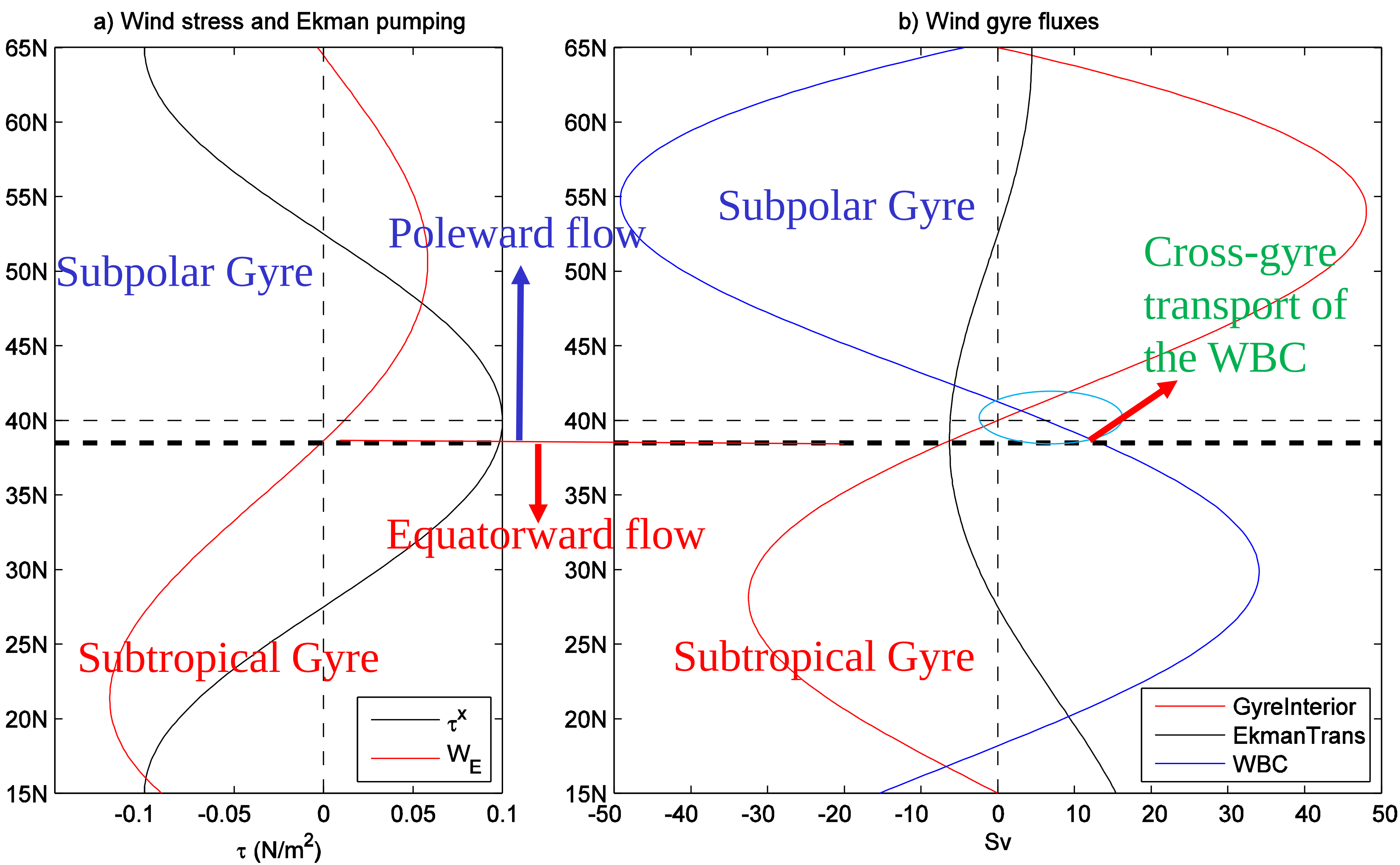


Inertial western boundary layer (No solution for the northern part of the western boundary layer)

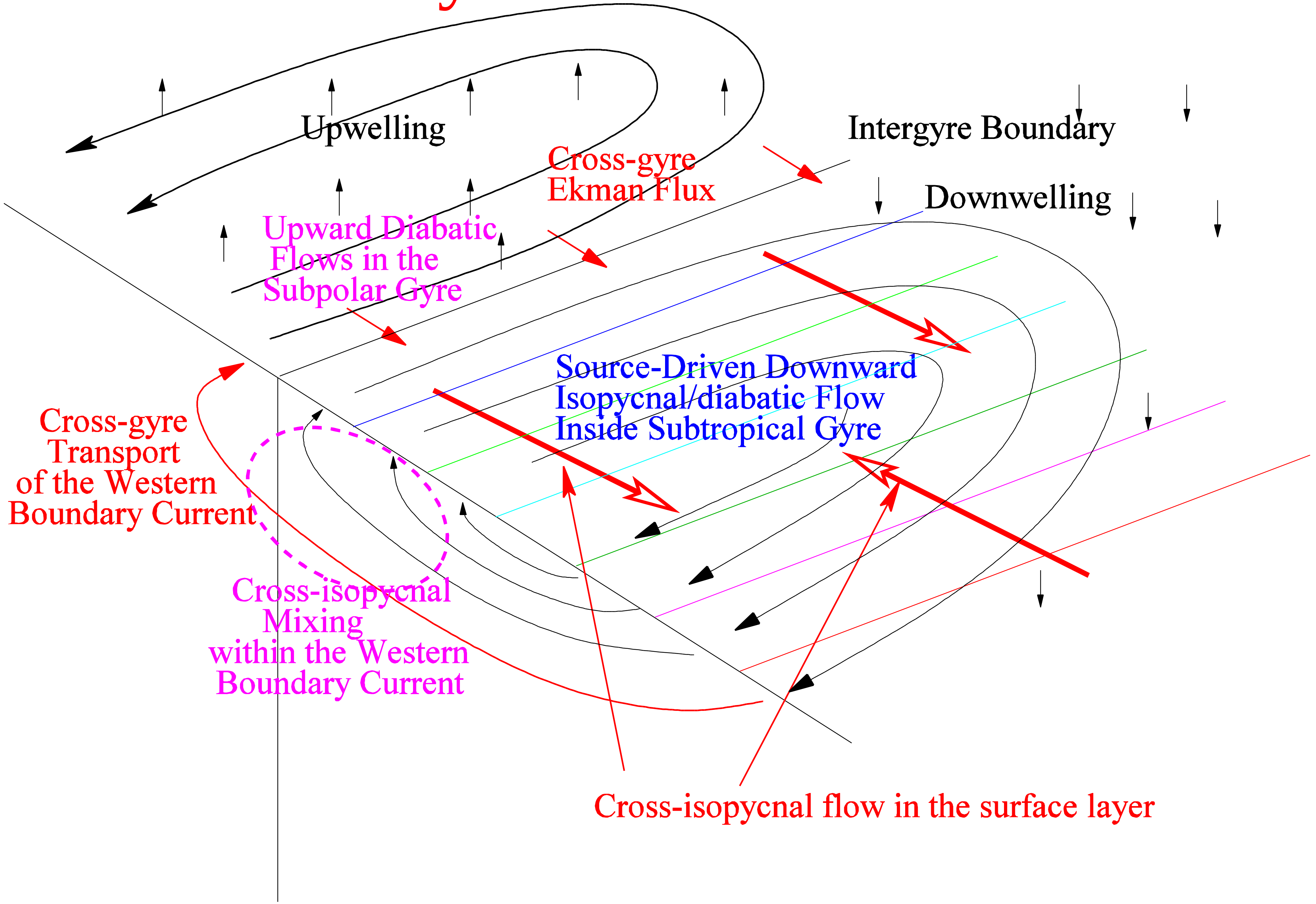
Mixing/dissipation required for close the budgets of the circulation



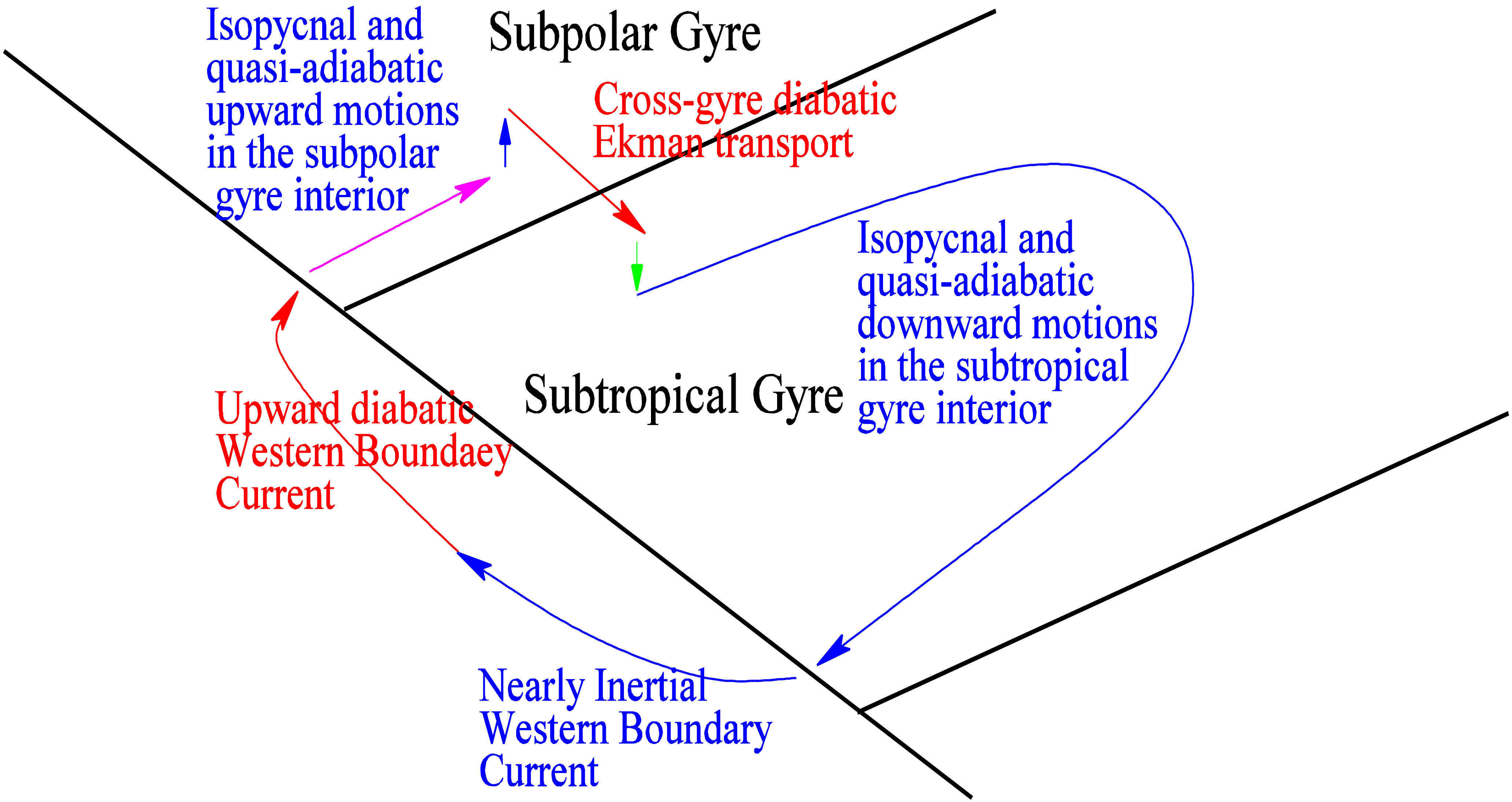
# Flow structure of a 2-gyre circulation



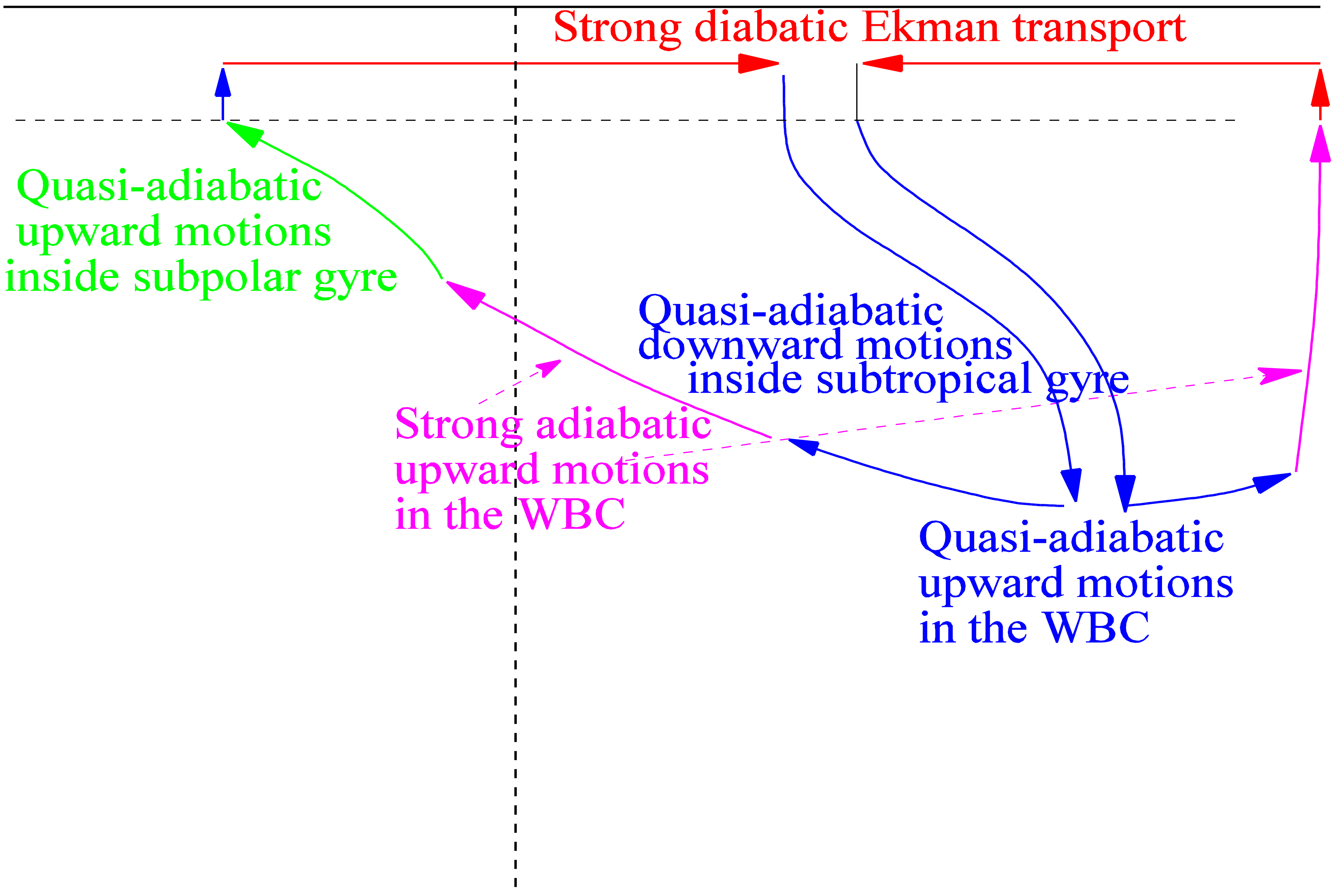
# 2-Gyre communication



# Typical trajectory of a water parcel



# An Eulerian view of the Ekman cells



# Lagrangian eddy flux?

Can Ekman pumping be cancelled by an upward Lagrangian eddy flux?

1) Flux in Eulerian and Lagrangian coordinates have different physical meaning --- they cannot be treated as the same fluxes

2) In the Lagrangian coordinates, there might be different fluxes associated with different tracers and different eddy sizes

3) Eddy flux associated with certain size of eddies may be compensated by a virtual flux of "eddy shadow" --- such a compensating virtual flux may move in the opposite direction.

4) When people in a meeting moves around, the air occupied by people should also move around --- the motions of air more or less compensates the flux associated with human being.